Math. Ann. 330, 759–807 (2004) **Mathematische Annalen**

DOI: 10.1007/s00208-004-0569-3

Techniques, computations, and conjectures for semi-topological K-theory

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Received: 11 February 2004 /
Published online: 24 August 2004 –

Abstract. We establish the existence of an “Atiyah-Hirzebruch-like” spectral sequence relating the morphic cohomology groups of a smooth, quasi-projective complex variety to its semi-topological K-groups. This spectral sequence is compatible with (and, indeed, is built from) the motivic spectral sequence that relates the motivic cohomology and algebraic K-theory of varieties, and it is also compatible with the classical Atiyah-Hirzebruch spectral sequence in algebraic topology. In the second part of this paper, we use this spectral sequence in conjunction with another computational tool that we introduce — namely, a variation on the integral weight filtration of the Borel-Moore (singular) homology of complex varieties introduced by H. Gillet and C. Soulé – to compute the semi-topological K-theory of a large class of varieties. In particular, we prove that for curves, surfaces, toric varieties, projective rational three-folds, and related varieties, the semi-topological K-groups and topological K-groups are isomorphic in all degrees permitted by cohomological considerations. We also formulate *integral* conjectures relating semi-topological K-theory to topological K-theory analogous to more familiar conjectures (namely, the Quillen-Lichtenbaum and Beilinson-Lichtenbaum Conjectures) concerning mod-n algebraic K-theory and motivic cohomology. In particular, we prove a local vanishing result for morphic cohomology which enables us to formulate precisely a conjectural identification of morphic cohomology by A. Suslin. Our computations verify that these conjectures hold for the list of varieties above.

Mathematics Subject Classification (2000): 19E20, 19E15, 14F43

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The first author was partially supported by the NSF and the NSA

The second author was supported by the Helen M. Galvin Fellowship of Northwestern University

The third author was partially supported by the NSF and the NSA

Contents

1. Introduction

The first and third authors have introduced and studied the (singular) semi-topological K-theory, $K_*^{sst}(X)$, of a complex variety X. Semi-topological K-theory lies part way between algebraic and topological K-theory in the sense that there are natural maps

$$K_q(X) \longrightarrow K_q^{sst}(X) \longrightarrow ku^{-q}(X^{an}), \quad q \geq 0,$$

that factor the natural map from the algebraic K-theory $K_*(X)$ of the variety X to the connective (complex) topological K-theory $ku^*(X^{an})$ of its underlying analytic space X^{an}. Semi-topological K-theory remains extremely difficult to compute, a fact which is not surprising given that the natural map $K_*(X, \mathbb{Z}/n) \to K_*^{sst}(X, \mathbb{Z}/n)$ is an isomorphism of graded rings for any positive integer n [16]. On the other hand, the semi-topological and topological K-theories of a point agree — i.e., we have isomorphisms $K_q^{sst}(\mathrm{Spec}\,\mathbb{C}) \cong ku^{-q}(pt)$ for all $q \geq 0$ — whereas each group $K_i(\mathrm{Spec}\,\mathbb{C})$ with $i > 0$ contains an uncountable rational vector space with little apparent structure. Moreover, for any smooth, quasi-projective complex variety X, the map of graded rings $K_*^{sst}(X) \to ku^{-*}(X^{an})$ becomes an isomorphism upon inverting the Bott element $\beta \in K_2^{sst}(\mathrm{Spec}\,\mathbb{C}) \cong ku^{-2}(pt) \cong \mathbb{Z}$ [16, 5.8]. We thus foresee the possibility that $K_*^{sst}(X)$ captures much of the complex algebraic geometry of X but discards the arithmetic data carried by $K_*(X)$.

The purpose of this paper is to introduce computational tools to facilitate the computation of $K_*^{sst}(X)$. As the interested reader will see below, we study the issue of when the natural map

$$K_*^{sst}(X) \to ku^{-*}(X^{an}) \tag{1.1}$$

is an isomorphism (or at least an isomorphism in certain degrees). In general, this map is rarely an isomorphism in all degrees — for example, if X has a non-trivial Griffiths group or if X has cohomology that is not algebraic, then (1.1) is not an isomorphism. However, we prove (1.1) is an isomorphism of graded rings in many of the cases in which such an isomorphism is conceivable (e.g., if X is a projective, smooth toric variety) — see Theorem 6.18. Such a result has geometric content, since $K_*^{sst}(X)$ for a projective variety X can be viewed as the homotopy groups of the stabilized mapping space of algebraic morphisms from X to Grassmannians viewed as a subspace of the stabilized mapping space of all continuous maps from X^{an} to Grassmannians. Thus, our various examples of projective varieties X satisfying $K_*^{sst}(X) \cong ku^{-*}(X^{an})$ reflect the remarkable fact that in these special cases the subspace of algebraic morphisms inside the space of continuous maps is a suitable homotopy-theoretic model.

Our main new technique for computing semi-topological K-theory is a spectral sequence

$$E_2^{p,q} = L^{-q}H^{p-q}(X) \Longrightarrow K_{-p-q}^{sst}(X)$$

relating the semi-topological K-theory of a smooth complex variety X to its morphic cohomology $L^*H^*(X)$. We further show that the natural maps

$$K_*(X) \longrightarrow K_*^{sst}(X) \longrightarrow ku^{-*}(X^{an})$$

can be realized as the maps on abutments of natural maps of spectral sequences from the motivic spectral sequence (cf. [6], [14], [22], [26], [33]) to the Atiyah-Hirzebruch spectral sequence [3]. Our construction also applies to provide maps of spectral sequences involving cohomology and K-theory with arbitrary coefficients, and we verify that the map from the motivic to the semi-topological spectral sequences with finite coefficients is an isomorphism. As in the motivic and topological contexts, our semi-topological spectral sequence degenerates rationally.

Our spectral sequence allows one to translate computations in morphic cohomology to computations in semi-topological K-theory. The other main computational technique developed in this paper is a method for computing morphic cohomology of smooth varieties — i.e., the E_2-terms of the semi-topological spectral sequence. Namely, for smooth varieties, morphic cohomology is naturally dual to Lawson homology, $L_*H_*(X)$, (cf. [11], [9]) which in turn maps naturally to (singular) Borel-Moore homology $H_n^{BM}(X^{an})$ via a generalized cycle map. We investigate the behavior of this map with respect to the (integral) weight filtration on $H_*^{BM}(X^{an})$ introduced by H. Gillet and C. Soulé [21] refining work of P. Deligne [8]. We show that the generalized cycle map factors as

$$L_rH_n(X) \to \tilde{W}_{-2r}H_n^{BM}(X) \to W_{-2r}H_n^{BM}(X^{an}) \subset H_n^{BM}(X^{an}),$$

where $W_*H_n^{BM}(X^{an})$ denotes the Deligne-Gillet-Soulé filtration and $\tilde{W}_*H_n^{BM}(X)$ is a related construction of our own invention. This factorization facilitates computations, for it is much easier to establish isomorphisms (relating Lawson homology and $\tilde{W}_*H_*^{BM}(X)$) than injections.

As mentioned above, semi-topological K-theory with finite coefficients is naturally isomorphic to algebraic K-theory with finite coefficients. There are various important conjectures for algebraic K-theory with finite coefficients which imply computational results. In this paper we also show that these conjectures have natural *integral* generalizations which seem somewhat plausible. In particular, we provide a local (for the Zariski topology) vanishing result which enables us to formulate precisely a conjectured identification due to A. Suslin of morphic cohomology which parallels a well known conjecture of Beilinson for motivic cohomology with finite coefficients. As we see, this has interesting computational consequences for semi-topological K-theory.

The approach we take in this paper is a continuation of [16] which established that the Chern character from semi-topological K-theory of a smooth variety to rational morphic cohomology is a rational isomorphism. The key tool of that paper was a recognition principle inspired by earlier work of V. Voevodsky which applies to the singular topological complex associated to a contravariant functor on complex varieties. As seen in [16], not only semi-topological K-theory but also Lawson homology and morphic cohomology can be formulated using such singular topological complexes. In this paper, we repeatedly use this formulation and the recognition principle of [16, 2.7] to enable computations.

We conclude this introduction by highlighting various results to be found in this paper.

Sections 2 through 4 concern the construction and properties of the semi-topological spectral sequence. Specifically, applying the technology of singular semi-topological complexes developed in [16] to the tower of spectra considered in [14], we establish in Section 2 the existence of a semi-topological spectral sequence relating morphic cohomology and semi-topological K-theory:

Theorem 1.2 (see Theorem 2.10). *For any smooth, quasi-projective complex variety X and abelian group A, there is a strongly convergent spectral sequence of the form*

$$E_2^{p,q} = L^{-q}H^{p-q}(X; A) \Longrightarrow K^{sst}_{-p-q}(X; A),$$

which is natural for morphisms of smooth varieties.

In Section 3, we prove this spectral sequence is compatible with both the motivic spectral sequence (connecting motivic cohomology and algebraic K-theory) and the classical Atiyah-Hirzebruch spectral sequence (connecting singular cohomology and topological K-theory). Specifically, we prove:

Theorem 1.3 (see Theorem 3.6). *For any smooth, quasi-projective complex variety X and any abelian group A, there are natural maps of strongly convergent spectral sequences*

$$\begin{array}{c} E_2^{p,q}(alg) = H^{p-q}_{\mathcal{M}}(X, A(-q)) \Longrightarrow K^{alg}_{-p-q}(X; A) \\ \downarrow \\ E_2^{p,q}(sst) = L^{-q}H^{p-q}(X; A) \Longrightarrow K^{sst}_{-p-q}(X; A) \\ \downarrow \\ E_2^{p,q}(top) = H^{p-q}(X^{an}; A) \Longrightarrow ku^{p+q}(X^{an}; A) \end{array}$$

inducing the usual maps on both E_2-terms and abutments.

In Section 4, we build upon the arguments of [16] to prove the semi-topological spectral sequence degenerates rationally, as expected:

Theorem 1.4 (see Theorem 4.2). *For a smooth, quasi-projective complex variety X, the semi-topological spectral sequence*

$$L^{-q}H^{p-q}(X) \Longrightarrow K^{sst}_{-p-q}(X)$$

degenerates rationally, and moreover this degeneration is induced by the semi-topological Chern character.

Sections 5 and 6 concern computations in semi-topological K-theory. The primary tools used in these computations are the semi-topological spectral sequence and a refinement of the integral weight filtration of singular (Borel-Moore) homology defined by Gillet-Soulé [21]. The latter refers to functors $\tilde{W}_t H_n^{BM}(-)$ we construct on the category of quasi-projective complex varieties that map surjectively to the weight filtration $W_t H_n^{BM}$ of Borel-Moore homology considered by Gillet-Soulé. In Section 5 we show the functors $\tilde{W}_t H_n^{BM}(-)$ enjoy numerous desirable properties, including a version of the projective bundle formula (see Theorem 5.14), and that they factor the map from Lawson homology to Borel-Moore homology in the following sense:

Theorem 1.5 (see Theorem 5.12). *For any quasi-projective complex variety U, the map from Lawson homology to Borel-Moore homology factors as*

$$L_t H_n(U) \to \tilde{W}_{-2t} H_n^{BM}(U) \to H_n^{BM}(U^{an}), \quad \text{for all } n, t \in \mathbb{Z}.$$

Moreover, the map $L_t H_n(U) \to \tilde{W}_{-2t} H_n^{BM}(U)$ is covariantly natural for proper morphisms and contravariantly natural for open immersions, for all $t, n \in \mathbb{Z}$. Also, the map $L_t H_* \to \tilde{W}_{-2t} H_*^{BM}$ is compatible with localization sequences.

From this we deduce immediately that the image of the generalized cycle map from Lawson homology $L_t H_*$ to Borel-Moore homology H_*^{BM} has weight at most $-2t$:

Corollary 1.6 (see Corollary 5.13). *For any quasi-projective complex variety U, the image of the canonical map $L_t H_n(U) \to H_n^{BM}(U^{an})$ lies in the part of weight at most $-2t$, $W_{-2t} H_n^{BM}(U^{an}) \subset H_n^{BM}(U^{an})$, of Borel-Moore homology.*

In Section 6 we apply Theorem 1.5 and the semi-topological spectral sequence (Theorem 1.2) to compute the semi-topological K-theory for a certain class of varieties. Coupled with a result on surfaces proved in Section 3, we obtain the following theorem:

Theorem 1.7 (see Theorem 7.14 and Proposition 6.19). *Let X be one of the following complex varieties:*

(1) *A smooth quasi-projective curve.*
(2) *A smooth quasi-projective surface.*
(3) *A smooth projective rational three-fold.*
(4) *A smooth projective rational four-fold*
(5) *A smooth quasi-projective linear variety (e.g., a smooth quasi-projective toric variety).*
(6) *A smooth toric fibration over a variety of type (1), (3), or (5) or over a smooth, quasi-projective surface having a smooth compactification such that all of H^2 is algebraic.*

Then the natural map $K_n^{sst}(X) \to ku^{-n}(X^{an})$ is an isomorphism for $n \geq \dim(X)-1$ and a monomorphism for $n = \dim(X) - 2$.

In the final section, we present a few conjectures which can be seen as integral forms of familiar conjectures in algebraic K-theory. Moreover, in Theorem 7.3 we prove a local vanishing result for morphic cohomology which enables us to precisely formulate a conjectured identification of morphic cohomology suggested by Suslin.

In a forthcoming paper, we intend to extend many of the results of this paper to real algebraic varieties, using the real semi-topological K-theory of [18].

We thank Andrei Suslin for many helpful conversations.

2. Construction of the semi-topological spectral sequence

The main result of [14] establishes the existence of a spectral sequence converging to the algebraic K-groups of a smooth, quasi-projective k-variety X whose E_2- terms are the motivic cohomology groups of X. In this section, we define a semi-topological analogue of this spectral sequence (when $k = \mathbb{C}$) and show that it receives a natural map from the motivic spectral sequence of [14].

We begin this section by briefly summarizing the construction of the motivic spectral sequence.

For the purposes of this paper, a *spectrum* means a prespectrum of Kan complexes (i.e., a sequence of Kan complexes $X^0, X^1, \ldots$ joined by "bonding maps" $X^i \to \Omega X^{i+1}$), an Ω-spectrum is what is sometimes called a "weak" Ω-spectrum (i.e., the bonding maps are homotopy equivalences), and a map of spectra is a *strict* map (i.e., we require maps to commute strictly with the bonding maps). An Ω-spectrum X is *n-connected*, for some integer n, if X^i is $(n+i)$-connected for each $i \geq 0$.

For any field k, Friedlander and Suslin construct contravariant functors $\mathcal{K}^{(q)}$ and $\mathcal{M}^{(q)}$, for each $q \geq 0$, from Sm/k to the category of (-1)-connected Ω-spectra, together with a family of sequences of natural transformations of the form

$$\mathcal{K}^{(q+1)}(X) \to \mathcal{K}^{(q)}(X) \to \mathcal{M}^{(q)}(X),$$

for each $q \geq 0$. (Thus, $\mathcal{K}^{(q)}(X)$ consists of a sequence of spaces $\mathcal{K}^{(q)}(X)^0, \mathcal{K}^{(q)}(X)^1, \ldots$ together with bonding maps, and similarly for $\mathcal{M}^{(q)}(X)$.) These functors and natural transformations are easily seen to extend to all schemes over k and to satisfy the following properties:

(1) All of the maps are natural on the category Sch/k.
(2) For any $X \in Sch/k$ and each $q \geq 0$, the composition

$$\mathcal{K}^{(q+1)}(X) \to \mathcal{K}^{(q)}(X) \to \mathcal{M}^{(q)}(X)$$

is the constant map.
(3) If X is smooth, then the sequence

$$\mathcal{K}^{(q+1)}(X) \to \mathcal{K}^{(q)}(X) \to \mathcal{M}^{(q)}(X)$$

is a weak homotopy fibration sequence of (-1)-connected Ω-spectra.
(4) There is a natural map $\mathcal{K}(X) \to \mathcal{K}^{(0)}(X)$ defined for $X \in Sch/k$ which is a weak homotopy equivalence whenever X is smooth. Here, $\mathcal{K}(X)$ denotes the usual K-theory Ω-spectrum of X as defined in [37].
(5) For X smooth and any abelian group A, we have natural isomorphisms

$$\pi_n \mathcal{M}^{(q)}(X; A) \cong H^{2q-n}_{\mathcal{M}}(X, A(q)),$$

for all $n, q \geq 0$, where $H^*_{\mathcal{M}}(X, A(*))$ denotes the motivic cohomology groups of X as defined, for example, in [15].
(6) For X smooth of dimension d, we have $\pi_n \mathcal{K}^{(q)}(X) = 0$ for $q > n + d$.

To construct the motivic spectral sequence for a smooth variety X, one sets

$$D_2^{p,q} = \pi_{-p-q}\mathcal{K}^{(-q)}(X) \quad \text{and} \quad E_2^{p,q} = \pi_{-p-q}\mathcal{M}^{(-q)}.$$

The above properties imply that these D_2- and E_2-terms form an exact couple, and moreover we have a natural, strongly convergent spectral sequence of the form

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X). \tag{2.1}$$

We obtain the semi-topological version of this spectral sequence by forming the semi-topological analogues of the functors $\mathcal{K}^{(q)}$ and $\mathcal{M}^{(q)}$.

Notation 2.2. We write $\underline{CW}$ for the category of spaces homeomorphic to finite dimensional CW-complexes endowed with the usual topology given by open subspaces and $\epsilon : \underline{CW} \to (Sch/\mathbb{C})_{Zar}$ for the morphism of sites given by the functor $U \mapsto U^{an}$ taking a complex variety to its associated analytic space and morphisms of complex varieties to their associated continuous maps. If $\mathcal{F}$ is a presheaf of sets, simplicial sets, or spectra on $Sch/\mathbb{C}$, we let $\epsilon^{\#}\mathcal{F}$ denote the presheaf on $\underline{CW}$ induced by the usual Kan extension formula

$$\epsilon^{\#}\mathcal{F}(T) = \varinjlim_{T \to U^{an}} \mathcal{F}(U).$$

Here, the limit is indexed by the opposite of the category $(Var^T)_{\mathbb{C}}$, whose objects are continuous maps $T \to U^{an}$ and in which a morphism from $T \to U^{an}$ to $T \to V^{an}$ is a morphism of varieties $U \to V$ causing the evident triangle to commute.

Frequently, we suppress the notation $\epsilon^{\#}$, so that given a presheaf $\mathcal{F}$ defined on $Sch/\mathbb{C}$ and a space T, we write simply $\mathcal{F}(T)$ for $\epsilon^{\#}\mathcal{F}(T)$. Given a scheme X and a space T we also write $\mathcal{F}(X \times T)$ for $\epsilon^{\#}(\mathcal{F}(X \times -))(T)$.

For any presheaf of simplicial sets $\mathcal{E}$ on $Sch/\mathbb{C}$, we write $\mathcal{E}^{sst}$ for the presheaf of simplicial sets given by

$$X \mapsto \operatorname{diag}(d \mapsto \mathcal{E}(X \times \Delta^d_{top})).$$

If $\mathcal{E}$ is a presheaf of Ω-spectra on $Sch/\mathbb{C}$, we write $\mathcal{E}^{sst}$ for the presheaf of spectra whose value at X is the spectrum defined by

$$\mathcal{E}^{sst}(X)^i = \operatorname{Ex}^\infty \operatorname{diag}(d \mapsto \mathcal{E}(X \times \Delta^d_{top})^i),$$

for $i \geq 0$, where Ex^∞ is the usual functor taking a general simplicial set to a homotopy equivalent Kan complex. The bonding maps for the spectrum $\mathcal{E}^{sst}(X)$ are defined by the composition of the canonical maps

$$\begin{aligned}\operatorname{Ex}^\infty \operatorname{diag}(d \mapsto \mathcal{E}(X \times \Delta^d_{top})^i) &\to \operatorname{Ex}^\infty \operatorname{diag}(d \mapsto \Omega\mathcal{E}(X \times \Delta^d_{top})^{i+1}) \\ &\to \operatorname{Ex}^\infty \Omega \operatorname{diag}(d \mapsto \mathcal{E}(X \times \Delta^d_{top})^{i+1}) \\ &\to \Omega \operatorname{Ex}^\infty \operatorname{diag}(d \mapsto \mathcal{E}(X \times \Delta^d_{top})^{i+1}) \quad (2.3)\end{aligned}$$

Remark 2.4.

(1) Observe that if $\mathcal{E}$ takes values in (-1)-connected Ω-spectra, then $\mathcal{E}^{sst}$ is an Ω-spectrum (and is (-1)-connected), since in this case the composition of the last two maps of (2.3) is a homotopy equivalence by [22, 7.1] and the first map in (2.3) is a homotopy equivalence in general.
(2) There is a natural transformation $id \to ()^{sst}$ (given by the augmentation of the cosimplicial space $\Delta^\bullet_{top}$) from the identity functor on presheaves of simplicial sets (or spectra) to the functor sending a presheaf $\mathcal{E}$ to the presheaf $\mathcal{E}^{sst}$.

Example 2.5. Let $\mathcal{K}$ be the (connective) algebraic K-theory spectrum as defined for example in [37]. Then we call $\mathcal{K}^{sst}$ the functor of *(singular) semi-topological K-theory*. That is, $\mathcal{K}^{sst}(X)$ is the Ω-spectrum with 0-th space

$$\mathcal{K}^{sst}(X)^0 = \operatorname{Ex}^\infty \operatorname{diag}(d \mapsto \mathcal{K}(X \times \Delta^d_{top})^0)$$

We build the semi-topological spectral sequence by building on the construction of the motivic spectral sequence, essentially by applying the functor $\mathcal{E} \mapsto \mathcal{E}^{sst}$ to the functors $\mathcal{K}^{(q)}$ and $\mathcal{M}^{(q)}$. The main technical result needed to facilitate this construction is the following one, which represents a slight generalization of a result proved by the first and third authors in [16]:

Theorem 2.6. *(See [16, 2.6]) Let $\theta : F \to G$ be a natural transformation of contravariant functors defined on $Sch/\mathbb{C}$ and taking values in the category of pointed Kan complexes. Assume additionally that for each $X \in Sch/\mathbb{C}$, $F(X)$ and $G(X)$ are homotopy commutative group-like H-spaces (i.e., pointed Kan complexes equipped with pairings that satisfy the axioms of an abelian group up to homotopy) and that for each morphism $X \to Y$ in $Sch/\mathbb{C}$, the maps $F(Y) \to F(X)$ and $G(Y) \to G(X)$ are H-maps (i.e., they commute with the pairings up to homotopy). Moreover, assume that the induced map $\theta : \pi_0 F(-) \to \pi_0 G(-)$ is a homomorphism of abelian groups. If the h-sheafification of the natural transformation of abelian groups $\theta : \pi_q F(-) \to \pi_q G(-)$ is an isomorphism for all $q \geq 0$, then*

$$\mathrm{diag}(d \mapsto F(\Delta^d_{top})) \to \mathrm{diag}(d \mapsto G(\Delta^d_{top}))$$

is a homotopy equivalence.

Proof. In [16, 2.6], this result is proven under the assumption that θ is actually a natural transformation of functors taking values in H-spaces, by which is meant $\theta : F(X) \to G(X)$ is an H-map for all $X \in Sch/\mathbb{C}$. However, the proof actually only needs the slightly weaker assumptions given here. Indeed, under these assumptions, we obtain a map of convergent spectral sequences from

$$\pi_p |d \mapsto \pi_q F(\Delta^d_{top})| \Longrightarrow \pi_{p+q} F(\Delta^\bullet_{top})$$

to

$$\pi_p |d \mapsto \pi_q G(\Delta^d_{top})| \Longrightarrow \pi_{p+q} G(\Delta^\bullet_{top})$$

which is an isomorphism on E_2-terms by [16, 2.3]. □

Corollary 2.7. *Suppose*

$$F \to G \to H$$

is a sequence of natural transformations of presheaves of Ω-spectra defined on $Sch/\mathbb{C}$. Assume H is N-connected for some (possibly negative) integer N. Suppose that for each $U \in Sch/\mathbb{C}$ the composition of $F(U) \to G(U) \to H(U)$ is the constant map and for each $U \in Sm/\mathbb{C}$ the sequence $F(U) \to G(U) \to H(U)$ is a weak homotopy fibration sequence of spectra. Then for any $X \in Sm/\mathbb{C}$, the sequence

$$F^{sst}(X) \to G^{sst}(X) \to H^{sst}(X)$$

is a weak homotopy fibration sequence of Ω-spectra.

Proof. Fix an $X \in Sm/\mathbb{C}$. For any $U \in Sch/\mathbb{C}$, define $L(U)$ to be the homotopy fiber of $G(U \times X) \to H(U \times X)$ so that L is also a presheaf of Ω-spectra on $Sch/\mathbb{C}$. By our hypothesis, there is a natural transformation of presheaves $F(- \times X) \to L$ on $Sch/\mathbb{C}$ which is a weak homotopy equivalence on each $U \in Sm/\mathbb{C}$. Observe that for all i, $F(U \times X)^i$ and $L(U)^i$ are clearly homotopy commutative group-like H-spaces, and thus Theorem 2.6 applies to show $\operatorname{diag} F(\Delta^\bullet_{top} \times X)^i \to \operatorname{diag} L(\Delta^\bullet_{top})^i$ is a homotopy equivalence. Hence $F^{sst}(X) \to L^{sst}(\operatorname{Spec}\mathbb{C})$ is a weak homotopy equivalence of spectra.

For each $i \geq 0$, the sequence

$$\operatorname{diag} L(\Delta^\bullet_{top})^i \to \operatorname{diag} G(\Delta^\bullet_{top} \times X)^i \to \operatorname{diag} H(\Delta^\bullet_{top} \times X)^i \qquad (2.8)$$

is obtained by taking diagonals of a degree-wise homotopy fibration sequence of bisimplicial sets. For $i > N + 1$, the Kan complex $H(\Delta^d_{top} \times X)^i$ is simply connected for each d, and thus by [7, B.4] we have that (2.8) is a homotopy fibration sequence of Kan complexes for such i. Consequently $L^{sst}(\operatorname{Spec}\mathbb{C}) = \operatorname{diag} L(\Delta^\bullet_{top})$ is weakly homotopy equivalent to the homotopy fiber of $G^{sst}(X) \to H^{sst}(X)$. □

Theorem 2.9. *If X is a smooth, quasi-projective complex variety, then for each $q \geq 0$, the sequence of maps*

$$\mathcal{K}^{(q+1),sst}(X) \to \mathcal{K}^{(q),sst}(X) \to \mathcal{M}^{(q),sst}(X)$$

is a weak homotopy fibration sequence of (-1)-connected Ω-spectra. Moreover, we have $\pi_n\mathcal{K}^{(q),sst}(X) = 0$ for $q > n + \dim(X)$.

Proof. By Corollary 2.7, the displayed sequence is a weak homotopy fibration sequence of Ω-spectra, and each is (-1)-connected by construction.

Recall that for any smooth F-variety U, where F is any ground field, we have $\pi_n\mathcal{K}^{(q)}(U) = 0$ whenever $q > n + \dim(U)$. In particular, for any field extension $F/\mathbb{C}$, we have $\pi_n\mathcal{K}^{(q)}(X \times \operatorname{Spec} F) = 0$ for $q > n + \dim(X)$. Moreover, for each n, the presheaf

$$G_n : U \mapsto \pi_n\mathcal{K}^{(q)}(X \times U)$$

defines a homotopy invariant pseudo-pretheory on $Sm/\mathbb{C}$ (cf. [14, 11.4]), and hence for any smooth, complex variety U and closed point $u \in U$, the map $G_n(\operatorname{Spec}\mathcal{O}_{U,u}) \to G_n(\operatorname{Spec} F)$ is an injection, where F denotes the field of fractions of $\mathcal{O}_{U,u}$. Hence, if $q > n + \dim(X)$, the Zariski sheafification of G_n vanishes on smooth schemes and thus, by resolution of singularities and Theorem 2.6, we have that $d \mapsto \pi_n\mathcal{K}^{(q)}(X \times \Delta^d_{top})$ is contractible. An application of the Bousfield-Friedlander spectral sequence [7, B.5]

$$\pi_s|d \mapsto \pi_t\mathcal{K}^{(q)}(X \times \Delta^d_{top})| \Longrightarrow \pi_{s+t}\mathcal{K}^{(q)}(X \times \Delta^\bullet_{top})$$

completes the proof. □

We can now immediately conclude the main theorem of this section:

Theorem 2.10. *For any smooth, quasi-projective complex variety X and abelian group A, there is a strongly convergent spectral sequence of the form*

$$E_2^{p,q} = L^{-q}H^{p-q}(X; A) \Longrightarrow K^{sst}_{-p-q}(X; A),$$

which is natural for morphisms of smooth varieties.

Proof. By setting $E_2^{p,q} = \pi_{-p-q}(\mathcal{M}^{(-q),sst}, A)$ and $D_2^{p,q} = \pi_{-p-q}(\mathcal{K}^{(-q),sst}, A)$, we obtain a strongly convergent spectral sequence

$$E_2^{p,q} \Longrightarrow \pi_{-p-q}\mathcal{K}^{(0),sst}(X; A).$$

Since $\mathcal{K}(X \times -) \to \mathcal{K}^{(0)}(X \times -)$ is a homotopy equivalence on smooth varieties, we have $\mathcal{K}^{(0),sst}(X)$ is homotopy equivalent to $\mathcal{K}^{sst}(X)$ by Theorem 2.6. It remains to identify the E_2-terms. This is done in [16, 3.5]. □

Proposition 2.11. *If X is a smooth, quasi-projective complex variety and A is an abelian group, there is a natural map of spectral sequences from the motivic spectral sequence*

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, A(-q)) \Longrightarrow K_{-p-q}(X, A)$$

to the semi-topological spectral sequence of Theorem 2.10 such that the map on abutments is the canonical map from algebraic K-theory to (singular) semi-topological K-theory. Moreover, if A is a finite abelian group, then this map is an isomorphism of spectral sequences.

Proof. The map of spectral sequences is obtained by applying the natural transformation of Remark 2.4 to the tower of spectra defining the motivic spectral sequence. If A is finite, the maps of E_2-terms are isomorphisms, since motivic and morphic cohomology agree for finite coefficients by [34]. □

3. Comparison of spectral sequences

In this section, we define a map from the semi-topological spectral sequence of Theorem 2.10 to the usual Atiyah-Hirzebruch spectral sequence. We do this by showing that a certain "topological realization" functor (that presumably is closely related to the one considered by Morel and Voevodsky in [30]) applied to the tower defining the motivic spectral sequence gives a model for the Postnikov tower of **bu**, and that there is a natural transformation from the semi-topological tower of Theorem 2.9 to this topological tower.

Notation 3.1. Suppose $\mathcal{E}$ is a presheaf of simplicial sets on $Sch/\mathbb{C}$. Define $\mathcal{E}^{top}$ to be the presheaf of simplicial sets defined on $\underline{CW}$ by the formula

$$D \mapsto \underline{\mathrm{Hom}}_{s.sets}(\mathrm{Sing}(D), \mathrm{Ex}^{\infty} \operatorname{diag} \mathcal{E}(\Delta^{\bullet}_{top})),$$

If $\mathcal{E}$ is a presheaf of spectra on $Sch/\mathbb{C}$, define $\mathcal{E}^{top}$ to be the presheaf of spectra on $\underline{CW}$ defined by

$$\mathcal{E}^{top}(D)^i = \underline{\mathrm{Hom}}_{s.sets}(\mathrm{Sing}(D), \mathrm{Ex}^{\infty} \operatorname{diag} \mathcal{E}(\Delta^{\bullet}_{top})^i),$$

with the bonding maps maps given in the usual manner.

We then have the following result:

Lemma 3.2. *If $\mathcal{E}$ is a presheaf of simplicial sets (resp., spectra) on $Sch/\mathbb{C}$, then for any $X \in Sch/\mathbb{C}$, there is a map of simplicial sets (resp., spectra)*

$$\mathcal{E}^{sst}(X) \to \mathcal{E}^{top}(X^{an})$$

natural in both X and $\mathcal{E}$ and satisfying the following properties.

(1) *If $\mathcal{E}$ is the presheaf of simplicial sets defined by* $\mathrm{Hom}(- \times \Delta^\bullet, Y)$ *for some variety Y, then $\mathcal{E}^{sst}(X) \to \mathcal{E}^{top}(X^{an})$ fits into the commutative square*

$$\begin{array}{ccc} \mathcal{E}^{sst}(X) & \longrightarrow & \mathcal{E}^{top}(X^{an}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(X \times \Delta^\bullet_{top}, Y) & \longrightarrow & \underline{\mathrm{Hom}}_{s.set}(\mathrm{Sing}(X^{an}), \mathrm{Sing}(Y^{an})) \end{array}$$

in which the vertical maps are induced by the "projection" maps $\Delta^d \times \Delta^d_{top} \to \Delta^d_{top}$. Note that these vertical maps are homology equivalences by [17, 1.2].

(2) *For any $\mathcal{E}$, the map*

$$\mathcal{E}^{sst}(\mathrm{Spec}\,\mathbb{C}) \to \mathcal{E}^{top}(pt)$$

is a weak homotopy equivalence.

Proof. We construct the natural transformation as follows (for $\mathcal{E}$ a presheaf of simplicial sets): Let X be a scheme. The map we want corresponds, by adjointness, to the map of simplicial sets

$$\mathcal{E}^{sst}(X) \times \mathrm{Sing}(X^{an}) \to \mathrm{Ex}^\infty \operatorname{diag} \mathcal{E}(\Delta^\bullet_{top})$$

which we obtain by composing the natural map $\operatorname{diag} \mathcal{E}(\Delta^\bullet_{top}) \to \mathrm{Ex}^\infty \operatorname{diag} \mathcal{E}(\Delta^\bullet_{top})$ with the map sending a d-simplex

$$((f : \Delta^d_{top} \to U^{an}, \alpha \in \mathcal{E}_d(X \times U)), g : \Delta^d_{top} \to X^{an})$$

of $\mathcal{E}^{sst}(X) \times \mathrm{Sing}(X^{an})$ to the d-simplex

$$((g, f) : \Delta^d_{top} \to X^{an} \times U^{an}, \alpha \in \mathcal{E}_d(X \times U))$$

of $\operatorname{diag} \mathcal{E}(\Delta^\bullet_{top})$. For a presheaf of spectra, we use this map on each level. The two required properties are immediate from the definitions. □

We apply the map of Lemma 3.2 when $\mathcal{E} = \mathcal{K}^{(q)}$ or $\mathcal{E} = \mathcal{M}^{(q)}$ for $q \geq 0$. In the case $q = 0$, we obtain the "standard" map from semi-topological K-theory to topological K-theory.

Proposition 3.3. *Let $\mathcal{K}$ denote the functor from $Sch/\mathbb{C}$ to spectra defining algebraic K-theory. Then for any $T \in \underline{CW}$, the spectrum $\mathcal{K}^{top}(T)$ is naturally homotopy equivalent to usual connective topological K-theory of T. Moreover, for any variety X, the map*

$$\mathcal{K}^{sst}(X) \to \mathcal{K}^{top}(X^{an})$$

of Lemma 3.2 is homotopic (on the level of total spaces) to the map constructed in [17, 2.0].

Proof. That $\mathcal{K}^{top}(T)$ gives the connective topological K-theory of T follows from the fact that $\mathcal{K}(\Delta^{\bullet}_{top})$ is homotopy equivalent to the spectrum **bu**, a fact which is established in [17].

The map from semi-topological K-theory to topological K-theory defined in [17] is constructed by using the following model for the functor $\mathcal{K}^{sst}$: Let Grass denote the ind-variety parameterizing the finite dimensional quotients of the countably infinite dimensional vector space $\mathbb{C}^{\infty}$. Then the simplicial set $\mathrm{Hom}(X \times \Delta^{\bullet}_{top}, \mathrm{Grass})$ admits an action by a certain E_{∞} operad, and hence one obtains an associated spectrum $\mathbb{S}\,\mathrm{Hom}(X \times \Delta^{\bullet}_{top}, \mathrm{Grass})$. It is proven in [17, 1.3] that there is a natural chain of homotopy equivalences joining the functor $\mathcal{K}^{sst}(-)$ (as we have defined it in this paper) to the functor $\mathbb{S}\,\mathrm{Hom}(- \times \Delta^{\bullet}_{top}, \mathrm{Grass})$. Similarly, it is proven in [29] that a model for connective topological K-theory of T is given by taking the spectrum associated to the space $\underline{\mathrm{Hom}}(T, \mathrm{Grass}^{an})$ equipped with the evident action by the linear isometries operad. Equivalently, the connective topological K-theory of T can be given by the spectrum associated to the simplicial set $\underline{\mathrm{Hom}}_{s.sets}(\mathrm{Sing}(T), \mathrm{Sing}(\mathrm{Grass}^{an}))$. The map from semi-topological K-theory to topological K-theory defined in [17] is induced by the canonical maps

$$\mathrm{Hom}(X \times \Delta^{d}_{top}, \mathrm{Grass}) \to \mathrm{Hom}_{s.sets}(\mathrm{Sing}(X) \times \Delta[d], \mathrm{Sing}(\mathrm{Grass}^{an})), \quad d \geq 0.$$

This is the bottom line of a diagram of the type in Lemma 3.2 (1), while the corresponding top line induces (after homotopy group completion) a model for the map defined in this paper. Applying the homotopy group completion functor makes the vertical maps homotopy equivalences, proving that the map of [17] is naturally homotopy equivalent to that defined in this paper. □

Theorem 3.4. *For each $q \geq 0$ and $D \in \underline{CW}$, the sequence*

$$\mathcal{K}^{(q+1),top}(D) \to \mathcal{K}^{(q),top}(D) \to \mathcal{M}^{(q),top}(D)$$

is a weak homotopy fibration sequence of Ω-spectra. Moreover, there are natural isomorphisms

$$\pi_n \mathcal{M}^{(q),top}(D) \cong H^{2q-n}(D, \mathbb{Z})$$

and

$$\pi_n \mathcal{K}^{(0),top}(D) \cong ku^{-n}(D),$$

for $n \in \mathbb{Z}$, so that we have a convergent spectral sequence of the form

$$E_2^{p,q}(D) = H^{p-q}(D) \Longrightarrow ku^{p+q}(D).$$

This spectral sequence is isomorphic to the classical Atiyah-Hirzebruch spectral sequence.

Proof. Since $\underline{\mathrm{Hom}}(\mathrm{Sing}(D), -)$ preserves weak homotopy fibration sequences of spectra, it suffices to prove that

$$\mathcal{K}^{(q+1),top}(pt) \to \mathcal{K}^{(q),top}(pt) \to \mathcal{M}^{(q),top}(pt) \tag{3.5}$$

is a weak homotopy fibration sequence for all $q \geq 0$, that $\mathcal{K}^{(0),top}(pt)$ is the usual spectrum **bu** representing connective topological K-theory, that $\mathcal{K}^{(q),top}(pt)$ is $(2q-1)$-connected, that $\mathcal{M}^{(q),top}(pt)$ is a $K(\mathbb{Z}, 2q)$, and that the map $\pi_{2q}\mathcal{K}^{(q),top}(pt) \to \pi_{2q}\mathcal{M}^{(q),top}(pt) = \mathbb{Z}$ is an isomorphism. For these conditions ensure that the collection of fibration sequences (3.5) coincide with those associated to the Postnikov tower of **bu**, and the classical Atiyah-Hirzebruch spectral sequence can be constructed by applying $\underline{\mathrm{Hom}}(\mathrm{Sing}(D), -)$ to this tower.

The claim that (3.5) is a weak homotopy fibration sequence of spectra follows from Theorem 2.9 using Lemma 3.2 (2). The fact that $\mathcal{K}^{(0),top}(pt)$ is homotopy equivalent to **bu** is given in the proof of Proposition 3.3. $\mathcal{M}^{(q),top}(pt)$ is a $K(\mathbb{Z}, 2q)$ since it is homotopy equivalent to (the singular complex of) the quotient topological abelian group $C_0(\mathbb{P}^q)^+/C_0(\mathbb{P}^{q-1})^+$ and hence to the infinite symmetric product of the $2q$-sphere. The remaining desired properties now follow. Indeed, if $\pi_n\mathcal{K}^{(q),top}(pt)$ did not vanish for a given q and $n < 2q$, then $\pi_n\mathcal{K}^{(t),top}(pt)$ would also not vanish for all $t \geq q$, since $\pi_n\mathcal{M}^{(t),top}(pt) = 0$ for all such t and since we have established the fibration sequences (3.5). This would contradict the vanishing of $\pi_n\mathcal{K}^{(t),top}(pt)$ for $t > n$, and hence $\mathcal{K}^{(q),top}(pt)$ must be $(2q-1)$-connected. The fact that the map $\pi_{2q}\mathcal{K}^{(q),top}(pt) \to \pi_{2q}\mathcal{M}^{(q),top}(pt) = \mathbb{Z}$ is an isomorphism now follows, using the long exact sequence associated to (3.5). □

We have now proven all the ingredients for the main result of this section.

Theorem 3.6. *For any smooth, quasi-projective complex variety X and any abelian group A, there are natural maps of strongly convergent spectral sequences*

$$\begin{array}{c} E_2^{p,q}(alg) = H_{\mathcal{M}}^{p-q}(X, A(-q)) \Longrightarrow K_{-p-q}^{alg}(X; A) \\ \downarrow \\ E_2^{p,q}(sst) = L^{-q}H^{p-q}(X; A) \Longrightarrow K_{-p-q}^{sst}(X; A) \\ \downarrow \\ E_2^{p,q}(top) = H^{p-q}(X^{an}; A) \Longrightarrow ku^{p+q}(X^{an}; A) \end{array}$$

inducing the usual maps on both E_2-terms and abutments.

Proof. The top vertical map is the one of Proposition 2.11, the bottom one is induced by the map of Lemma 3.2 using the identification of Theorem 3.4. □

As a first application of this comparison of spectral sequences we give a computation of the semi-topological K-theory of surfaces:

Theorem 3.7. *For any smooth quasi-projective surface S and abelian group A, the natural map*

$$K_n^{sst}(S, A) \to ku^{-n}(S^{an}, A)$$

is an isomorphism for $n \geq 1$ and a monomorphism for $n = 0$.

Proof. We claim that for any $r \geq 2$, the comparison map $E_r^{p,q}(sst) \to E_r^{p,q}(top)$ is an isomorphism for $p + q < 0$ and a monomorphism for $p + q = 0$. If $r = 2$, then in the case $q = 0$, these assertions are evident, in the case $q = -1$, they follow from a theorem of Friedlander and Lawson [12, 9.3], and in the case $q \leq -2$, they follow from duality relating morphic cohomology and Lawson homology [11] and the fact that $L_0H_n(S, A) \cong H_n^{BM}(S^{an}, A)$ for all n.

Proceed by induction on r, using the commutative diagram

$$\begin{array}{ccccc} E_r^{p-r,q+r-1}(sst) & \longrightarrow & E_r^{p,q}(sst) & \longrightarrow & E_r^{p+r,q-r+1}(sst) \\ \downarrow & & \downarrow & & \downarrow \\ E_r^{p-r,q+r-1}(top) & \longrightarrow & E_r^{p,q}(top) & \longrightarrow & E_r^{p+r,q-r+1}(top) \end{array}$$

formed by the E_r-differentials of the spectral sequences. If $p+q \leq 0$, then the left vertical map in this diagram is an isomorphism and the middle vertical map is a monomorphism, and hence a diagram chase shows that $E_{r+1}^{p,q}(sst) \to E_{r+1}^{p,q}(top)$ is a monomorphism. If $p + q < 0$, then we have in addition that the middle map is an isomorphism and the right map is a monomorphism, and another diagram chase shows that $E_{r+1}^{p,q}(sst) \to E_{r+1}^{p,q}(top)$ is surjective.

In particular, we conclude that the maps $E_\infty^{p,q}(sst) \to E_\infty^{p,q}(top)$ are isomorphisms for $p + q < 0$ and monomorphisms for $p + q = 0$. The result follows. □

4. Rational degeneration

In this section we prove that, just as with the motivic and topological spectral sequences, the semi-topological spectral sequence degenerates rationally. That is, we show that the map $D_2^{p,q}(sst)_{\mathbb{Q}} \to D_2^{p-1,q+1}(sst)_{\mathbb{Q}}$ is injective and, equivalently, the map $E_2^{p,q}(sst)_{\mathbb{Q}} \to D_2^{p+2,q-1}(sst)_{\mathbb{Q}}$ is the zero map, for all p, q. Here, for an abelian group A, we set $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Consequently, we have the rational isomorphism

$$K_n^{sst}(X)_{\mathbb{Q}} \cong \bigoplus_{q \geq 0} L^q H^{2q-n}(X; \mathbb{Q})$$

valid for any smooth, complex variety X. Such an isomorphism was first established in [16], where it is proven that the Chern character map determines a ring isomorphism of this form. We prove here that the rational degeneration of the spectral sequence corresponds to the Chern character isomorphism, in a sense made precise below. Moreover, since the semi-topological Chern character is compatible with the algebraic and topological Chern characters by [16, 4.7], this rational splitting is compatible with the rational splitting of the motivic and topological spectral sequences.

Remark 4.1. The motivic spectral sequence is shown to degenerate rationally by establishing the existence of Adams operations, ψ^k for $k \geq 1$, on the D_2 and E_2 terms of the motivic spectral sequence and then proving that $D_2^{p,q} \otimes \mathbb{Q}$ is a direct sum of summands having weight at least $-q$ while $E_2^{p,q} \otimes \mathbb{Q}$ is of pure weight $-q$. (An element has weight t if it is an eigenvector with eigenvalue k^t for the operator ψ^k, for any $k \geq 1$.) Thus the map $E_2^{p,q} \to D_2^{p+2,q-1}$ must be the zero map rationally. Presumably, one could define Adams operations on $K_n^{sst}(X)$ as well as the E_2- and D_2-terms of the semi-topological spectral sequence, but we have not attempted to do so in this paper.

Recall that the semi-topological Chern character map

$$ch^{sst} : \bigoplus_n K_n^{sst}(X) \to \bigoplus_{n,t} L^t H^{2t-n}(X, \mathbb{Q}),$$

defined in [16, §4], is a natural transformation of graded-ring-valued functors on the category of smooth varieties, and it induces a rational isomorphism.

Theorem 4.2. *Suppose X is a smooth, quasi-projective complex variety. Then the composition of*

$$\pi_n \mathcal{K}^{(t),sst}(X)_{\mathbb{Q}} \to K_n^{sst}(X)_{\mathbb{Q}} \stackrel{ch_n^{sst}}{\to} \bigoplus_{s \geq 0} L^s H^{2s-n}(X, \mathbb{Q}) \twoheadrightarrow \bigoplus_{s \geq t} L^s H^{2s-n}(X, \mathbb{Q}) \tag{4.3}$$

is an isomorphism for all $t, n \geq 0$. Consequently, the semi-topological spectral sequence

$$L^{-q} H^{p-q}(X) \Longrightarrow K_{-p-q}^{sst}(X)$$

degenerates rationally.

Proof. For $n = 0$, the sequence of maps (4.3) is a quotient of the corresponding algebraic sequence

$$\pi_0 \mathcal{K}^{(t)}(X)_{\mathbb{Q}} \to K_0(X)_{\mathbb{Q}} \stackrel{ch}{\to} \bigoplus_{s \geq 0} H^{2s}_{\mathcal{M}}(X, \mathbb{Q}(s)) \twoheadrightarrow \bigoplus_{s \geq t} H^{2s}_{\mathcal{M}}(X, \mathbb{Q}(s)). \tag{4.4}$$

In fact, if F denotes $\pi_0\mathcal{K}^{(t)}(-)_{\mathbb{Q}}$, $K_0(-)_{\mathbb{Q}}$, or $H^{2s}_{\mathcal{M}}(-, \mathbb{Q}(s))$, and F^{sst} denotes its semi-topological counterpart, we have a natural exact sequence

$$\bigoplus_{(C,c_0,c_1)} F(X \times C) \to F(X) \to F^{sst}(X) \to 0$$

where (C, c_0, c_1) ranges over the collection of smooth, connected complex curves C with specified closed points c_0, c_1. The proof of [16, 1.10] shows that the composition of (4.4) is an isomorphism, and thus the composition of (4.3) is an isomorphism for $n = 0$.

To establish the result for all $n > 0$, note that since $\pi_n\mathcal{K}^{(t),sst}(X) = 0$ and $L^t H^{2n-t}(X) = 0$ for $t \gg n$, it suffices by using descending induction on t to establish that the sequence

$$0 \to \pi_n\mathcal{K}^{(t+1),sst}(X)_{\mathbb{Q}} \to \pi_n\mathcal{K}^{(t),sst}(X)_{\mathbb{Q}} \stackrel{ch^{sst}_{n,t}}{\to} L^t H^{2t-n}(X, \mathbb{Q}) \to 0$$

is exact. If $n > 0$, then for all s the map $ch^{sst}_{n,s}$ is a non-zero rational multiple of the Segre class map (defined in [16, 4.1])

$$s^{sst}_{n,s} : K^{sst}_n(X)_{\mathbb{Q}} \to L^s H^{2s-n}(X, \mathbb{Q}),$$

and thus it suffices to prove

$$0 \to \pi_n\mathcal{K}^{(t+1),sst}(X)_{\mathbb{Q}} \to \pi_n\mathcal{K}^{(t),sst}(X)_{\mathbb{Q}} \stackrel{s^{sst}_{n,t}}{\to} L^t H^{2t-n}(X, \mathbb{Q}) \to 0$$

is exact for all $n, t > 0$. (Observe that when $t = 0$, we have $\pi_n\mathcal{K}^{(1),sst}(X)_{\mathbb{Q}} \to \pi_n\mathcal{K}^{(0),sst}(X)_{\mathbb{Q}}$ is an isomorphism and $L^0 H^{2t-n}(X, \mathbb{Q}) = 0$ for all $n > 0$, and so exactness in this case is clear.) The exactness of this sequence will hold provided we show the composition of

$$\pi_n\mathcal{K}^{(t),sst}(X)_{\mathbb{Q}} \to K^{sst}_n(X)_{\mathbb{Q}} \stackrel{Seg}{\to} \bigoplus_{s \geq 1} L^s H^{2s-n}(X, \mathbb{Q}) \to \bigoplus_{s \geq t} L^s H^{2s-n}(X, \mathbb{Q}) \tag{4.5}$$

is an isomorphism, where Seg denotes the total Segre class map, given by $Seg = s_{n,1} + s_{n,2} + \cdots$.

To establish such an isomorphism, we will use Theorem 2.6 and the fact that the algebraic version of this result holds — namely, the composition of

$$\pi_n\mathcal{K}^{(t)}(X)_{\mathbb{Q}} \to K_n(X)_{\mathbb{Q}} \stackrel{Seg}{\to} \bigoplus_{s \geq 1} H^{2s-n}_{\mathcal{M}}(X, \mathbb{Q}(s)) \to \bigoplus_{s \geq t} H^{2s-n}_{\mathcal{M}}(X, \mathbb{Q}(s))$$

is an isomorphism for $n, t > 0$ as shown in the proof of [16, 1.10]. To employ Theorem 2.6, we must realize the map

$$\pi_n\mathcal{K}^{(t)}(X)_{\mathbb{Q}} \to \bigoplus_{s \geq t} L^s H^{2s-n}_{\mathcal{M}}(X, \mathbb{Q}(s))$$

as coming from a natural transformation of spaces. In [16, §1], it is shown that the total Segre class map arises as a natural transformation of functors

$$Seg : \mathcal{K}_{geom}(-) \to \mathcal{H}_{mult}(-),$$

where $\mathcal{K}_{geom}(-)$ is a functor defining the algebraic K-theory of varieties and $\mathcal{H}_{mult}(-)$ is a certain functor that fits into a natural homotopy fibration sequence of the form

$$\mathrm{Hom}(- \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^\infty))_1^+ \to \mathcal{H}_{mult}(-) \to \mathrm{Hom}(-, \mathbb{Z}),$$

Moreover, there is a natural homotopy equivalence

$$\mathrm{Hom}(- \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^\infty))_1^+ \xrightarrow{\sim} \varinjlim{}_N \prod_{t=1}^{N} \mathcal{H}_{\mathcal{M},naive}(-, \mathbb{Z}(t))$$

where

$$\mathcal{H}_{\mathcal{M},naive}(-, \mathbb{Z}(t)) = \mathrm{Hom}(- \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^t))^+ / \mathrm{Hom}(- \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^{t-1}))^+$$

is a functor from $Sch/\mathbb{C}$ to simplicial abelian groups whose homotopy groups are naturally isomorphic to the motivic groups for smooth varieties. A slight complication arises in that the functor $\mathcal{K}_{geom}(X)$ is not the same model for the algebraic K-theory space $\mathcal{K}(X)$ of a variety X used in this paper. (Specifically, $\mathcal{K}_{geom}(X)$ is given by the homotopy theoretic group completion of $\mathrm{Hom}(X \times \Delta^\bullet, \mathrm{Grass})$ where Grass parameterizes all finite dimensional subspaces of $\mathbb{C}^\infty$.) However, the functors $\mathcal{K}$ and $\mathcal{K}_{geom}$ are related by a "zig-zag" chain of natural maps that are weak homotopy equivalences on smooth varieties [23, 3.3], [19, 6.8]. Therefore, by taking homotopy pullbacks as necessary, we may form a sequence of natural maps

$$\cdots \to \mathcal{K}_{geom}^{(t+1)} \to \mathcal{K}_{geom}^{(t)} \to \cdots \to \mathcal{K}_{geom}^{(0)} = \mathcal{K}_{geom}$$

such that $\mathcal{K}_{geom}^{(t)}$, for all t, is related to $\mathcal{K}^{(t)}$ via a zig-zag chain of natural maps that are weak homotopy equivalences on smooth varieties. In particular, by Theorem 2.6, the spaces $\mathcal{K}^{(t)}(X \times \Delta_{top}^\bullet)$ and $\mathcal{K}_{geom}^{(t)}(X \times \Delta_{top}^\bullet)$ are homotopy equivalent, and it suffices to consider the map $\mathcal{K}_{geom}^{(t)} \to \mathcal{H}_{mult}$, defined to be the composition of $\mathcal{K}_{geom}^{(t)} \to \mathcal{K}_{geom} \xrightarrow{Seg} \mathcal{H}_{mult}$. Since the composition of $\mathcal{K}_{geom} \to \mathcal{H}_{mult} \to \mathrm{Hom}(-, \mathbb{Z})$ is the rank map, the composition $\mathcal{K}_{geom}^{(t)} \to \mathcal{H}_{mult} \to \mathrm{Hom}(-, \mathbb{Z})$ is the constant map with value 0 (since we are assuming $t > 0$). Therefore, there is a natural map from $\mathcal{K}_{geom}^{(t)}$ to the homotopy fiber of $\mathcal{H}_{mult} \to \mathrm{Hom}(-, \mathbb{Z})$ over 0. Let us denote this homotopy fiber as $\mathcal{F}_0$ — it is a functor from $Sch/\mathbb{C}$ to simplicial sets and there exists a natural zig-zag of weak homotopy equivalences joining it to the functor $\varinjlim{}_N \prod_{t=1}^{N} \mathcal{H}_{\mathcal{M},naive}(-, \mathbb{Z}(t))$. By taking homotopy pullback as necessary, we can thus form the diagram of natural transformations

$$\mathcal{K}_{geom}^{(t)} \xleftarrow{\sim} \mathcal{G} \to \varinjlim{}_N \prod_{s=1}^{N} \mathcal{H}_{\mathcal{M},naive}(-, \mathbb{Q}(s)) \to \varinjlim{}_N \prod_{s=t}^{N} \mathcal{H}_{\mathcal{M},naive}(-, \mathbb{Q}(s)),$$

for a suitable functor $\mathcal{G}$ from $Sch/\mathbb{C}$ to Kan complexes.

As mentioned above, the map

$$\mathcal{G}(X) \to \varinjlim_N \prod_{s=q}^{N} \mathcal{H}_{\mathcal{M},naive}(X, \mathbb{Q}(s))$$

is a rational homotopy equivalence for any smooth variety X as shown in the proof of [16, 1.10]. On π_0 groups, this map is isomorphic to the composition of

$$\pi_0\mathcal{K}^{(t)}(X) \to K_0(X) \stackrel{Seg}{\to} \left(\{1\} \times \bigoplus_{s\geq q} CH^s(X)_{\mathbb{Q}}\right)^{\times},$$

where the target is a group under intersection of cycles (i.e., cup product) — in particular, this map is a homomorphism. Finally, both $\mathcal{G}$ and $\varinjlim_N \prod_{s=q}^{N} \mathcal{H}_{\mathcal{M},naive}(X, \mathbb{Q}(s))$ take values in H-spaces, where the H-space structure on the latter is given by cup product for motivic cohomology. Thus, Theorem 2.6 applies to give that

$$\mathcal{G}(X \times \Delta^{\bullet}_{top}) \to \varinjlim_N \prod_{s=q}^{N} \mathcal{H}_{\mathcal{M},naive}(X \times \Delta^{\bullet}_{top}, \mathbb{Q}(s))$$

is a rational homotopy equivalence, and hence the composition of (4.5) is an isomorphism. □

5. Refined cycle map using weight filtrations

In this section we demonstrate that the canonical map from Lawson homology to integral Borel-Moore homology

$$L_t H_*(U) \to H_*^{BM}(U^{an})$$

factors through a refined homological invariant $\tilde{W}_{-2t} H_*^{BM}(U^{an})$ that is closely related to the weight filtration $W_* H_*^{BM}(U^{an})$ of $H_*^{BM}(U^{an})$ considered by Deligne [8] and Gillet-Soulé [21]. Here, U is an arbitrary (possibly singular) quasi-projective complex variety. This refined map allows one, in particular, to prove that the Lawson homology and singular homology of smooth, projective toric varieties are isomorphic. This and other consequences will be worked out in the next section.

We begin with the definition of the weight filtration on $H_*^{BM}(U^{an})$ for any quasi-projective complex variety U. This definition is due to Deligne [8] (for rational coefficients) and to Gillet-Soulé [21] (for arbitrary coefficients). To begin, one chooses a compactification $U \subset X$ of U so that X is a projective, complex variety, and sets Y to be the reduced closed complement of U in X. As shown in [21, 1.4], one may construct so-called "hyper-envelopes" $X_{\bullet} \to X$ and $Y_{\bullet} \to Y$ such that X_n and Y_n are smooth, projective varieties and such that there is a (proper) map of simplicial schemes $Y_\bullet \to X_\bullet$ extending the closed immersion $Y \to X$. A "hyper-envelope" is an augmented simplicial variety $X_\bullet \to X$ such that the canonical map from $X_n \to (\mathrm{cosk}_{n-1}(X_\bullet))_n$ is a proper map that is surjective on F-points for any field F. We will also refer to the map of simplicial varieties $Y_\bullet \to X_\bullet$ as a "hyper-envelope" of the map $Y \to X$. The existence of smooth hyper-envelopes follows readily from Hironaka's resolutions of singularities. Moreover, a hyper-envelope $X_\bullet \to X$ is precisely the same thing as a hyper-cover for the proper cdh topology [41, 2.17].

Let $\mathbb{Z}\,\mathrm{Sing}(-)$ denote the functor taking a space to the complex of singular chains (with boundary maps given by alternating sums of face maps). We often apply $\mathbb{Z}\,\mathrm{Sing}(-)$ to the analytic space X^{an} of a complex variety X, in which case we simply write $\mathbb{Z}\,\mathrm{Sing}(X)$ for $\mathbb{Z}\,\mathrm{Sing}(X^{an})$. Applying the functor $\mathbb{Z}\,\mathrm{Sing}(-)$ to a simplicial variety $X_\bullet$ degree-wise and taking alternating sums of the face maps of $X_\bullet$ results in the bicomplex

$$\cdots \to \mathbb{Z}\,\mathrm{Sing}(X_1) \to \mathbb{Z}\,\mathrm{Sing}(X_0).$$

Since $Y_\bullet \to X_\bullet$ is a map of simplicial schemes, we may form the bicomplex

$$\cdots \to \mathbb{Z}\,\mathrm{Sing}(X_2) \oplus \mathbb{Z}\,\mathrm{Sing}(Y_1) \to \mathbb{Z}\,\mathrm{Sing}(X_1) \oplus \mathbb{Z}\,\mathrm{Sing}(Y_0) \to \mathbb{Z}\,\mathrm{Sing}(X_0)$$

in the evident manner by taking the cone of the chain map

$$\mathbb{Z}\,\mathrm{Sing}_n(X_\bullet) \to \mathbb{Z}\,\mathrm{Sing}_n(Y_\bullet)$$

for each fixed n. To simplify notation, we set $U_n = X_n \coprod Y_{n-1}$ (with $Y_{-1} = \emptyset$) so that the above bicomplex becomes

$$\cdots \to \mathbb{Z}\,\mathrm{Sing}(U_1) \to \mathbb{Z}\,\mathrm{Sing}(U_0). \tag{5.1}$$

Abusing the notation of Gillet-Soulé a bit, we will call the bicomplex (5.1) the "singular weight complex" of U. As we show in Proposition 5.2, the homology of the total complex associated to the singular weight complex of U gives the Borel-Moore homology of U^{an}. In Proposition 5.9, it is asserted that this singular weight complex is independent of the choices made, up to canonical isomorphism in the derived category.

Recall that $Z_t(-)$ is the functor taking a quasi-projective complex variety X to the topological abelian group of t-dimensional cycles on X (for the exact definitions, see [10] and [27]). The functor $Z_t(-)$ is covariant for proper morphisms (via push-forward of cycles) and contravariant for open immersions (via restriction of cycles). We let $\mathrm{Sing}\, Z_t(-)$ denote the functor taking a variety U to the chain complex associated to the simplicial abelian group obtained by applying $\mathrm{Maps}(\Delta^n_{top}, -)$ to $Z_t(U)$. The Lawson homology groups are defined from $Z_t(-)$ via the formula

$$L_t H_n(X) = \pi_{n-2t} Z_t(X) = h_{n-2t}\, \mathrm{Sing}\, Z_t(X), \quad \text{if } t \geq 0.$$

It is convenient to extend this definition by setting

$$L_t H_n(X) = L_0 H_n(X), \quad \text{if } t < 0.$$

(Another common convention is to set $L_t H_n(X) = L_0 H_{n-2t}(X \times \mathbb{A}^{-t})$ for $t < 0$, but the canonical map given by composing flat pullback and the s-map, $L_0 H_n(X) \to L_{-t} H_{n-2t}(X \times \mathbb{A}^{-t}) \to L_0 H_{n-2t}(X \times \mathbb{A}^{-t})$, is an isomorphism for all $t \geq 0$.)

For a quasi-projective complex variety U, if we choose a compactification X with reduced complement Y and smooth hyper-envelope $Y_\bullet \to X_\bullet$ of the closed immersion $Y \rightarrowtail X$, then we can apply $\operatorname{Sing} Z_t(-)$ to each $U_i = X_i \coprod Y_{i-1}$ to obtain the bicomplex

$$\cdots \to \operatorname{Sing} Z_t(U_1) \to \operatorname{Sing} Z_t(U_0)$$

in a parallel fashion to the construction of the singular weight complex of U.

Proposition 5.2. *For any quasi-projective variety U, the canonical maps*

$$Tot\,(\cdots \to \mathbb{Z}\operatorname{Sing}(U_1) \to \mathbb{Z}\operatorname{Sing}(U_0)) \longrightarrow \mathbb{Z}\operatorname{Sing}(X)/\mathbb{Z}\operatorname{Sing}(Y)$$

and

$$Tot\,(\cdots \to \operatorname{Sing} Z_t(U_1) \to \operatorname{Sing} Z_t(U_0)) \longrightarrow \operatorname{Sing} Z_t(U), \quad \textit{for } t \geq 0,$$

are quasi-isomorphisms, where the varieties U_n, $n \geq 0$, and the maps between them are constructed as above. Thus we have natural isomorphisms

$$H_n^{BM}(U^{an}) \cong h_n Tot(\cdots \to \mathbb{Z}\operatorname{Sing}(U_1) \to \mathbb{Z}\operatorname{Sing}(U_0))$$

and

$$L_t H_n(U) \cong h_{n-2t} Tot(\cdots \to \operatorname{Sing} Z_t(U_1) \to \operatorname{Sing} Z_t(U_0)), \quad \textit{for } t \geq 0.$$

Proof. Since Lawson homology satisfies localization, the sequence

$$\operatorname{Sing} Z_t(Y) \to \operatorname{Sing} Z_t(X) \to \operatorname{Sing} Z_t(U)$$

is a distinguished triangle, and thus for the assertions involving Lawson homology, it suffices to prove

$$\operatorname{Sing} Z_t(X_\bullet) \to \operatorname{Sing} Z_t(X) \quad \text{and} \quad \operatorname{Sing} Z_t(Y_\bullet) \to \operatorname{Sing} Z_t(Y)$$

are quasi-isomorphisms. In fact, we prove $\operatorname{Sing} Z_t(X_\bullet) \to \operatorname{Sing} Z_t(X)$ is a quasi-isomorphism for any hyper-envelope $X_\bullet \to X$ with X a quasi-projective complex variety. Likewise, for the assertions involving Borel-Moore homology, it suffices to prove $\mathbb{Z}\operatorname{Sing}(X_\bullet) \to \mathbb{Z}\operatorname{Sing}(X)$ is a quasi-isomorphism for an arbitrary hyper-envelope.

Observe that it suffices to show $F^\bullet(X) \to F^\bullet(X_\bullet)$ is a quasi-isomorphism where $F^\bullet$ is either $\mathrm{Hom}(\mathbb{Z}\,\mathrm{Sing}_\bullet(-), \mathbb{Q}/\mathbb{Z})$ or $\mathrm{Hom}(\mathrm{Sing}_\bullet\, Z_t(-), \mathbb{Q}/\mathbb{Z})$. In each case, $F^\bullet$ is a bounded below cochain complex of presheaves of abelian groups on the category of quasi-projective varieties and proper morphisms. Following Voevodsky [41], we call such a complex "flasque" for the proper cdh topology (pcdh topology, for short) provided that

$$F^\bullet(X) \to F^\bullet(Z) \oplus F^\bullet(X') \to F^\bullet(Z')$$

is a distinguished triangle (more precisely, provided that the canonical map from the cone of $F^\bullet(X) \to F^\bullet(Z) \oplus F^\bullet(X')$ to $F^\bullet(Z')$ is a quasi-isomorphism), for every "abstract blow-up"

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X. \end{array}$$

(Recall that an abstract blow-up consists of a proper map $X' \to X$ and a closed immersion $Z \subset X$ such that the induced map $X' - Z' \to X - Z$ is an isomorphism, where Z' denotes the closed subscheme $Z \times_X X'$ of X'.) The presheaves we are interested in are flasque because Lawson homology and Borel-Moore homology satisfy localization. We shall prove $F^\bullet(X) \to F^\bullet(X_\bullet)$ is a quasi-isomorphism for any hyper-envelope $X_\bullet \to X$ for any $F^\bullet$ having these properties.

Let $F^\bullet \to I^\bullet$ be a resolution of $F^\bullet$ by a bounded below complex of injective pcdh sheaves — i.e., $F^\bullet \to I^\bullet$ is locally a quasi-isomorphism for the pcdh topology and each I^n is injective. The complex $I^\bullet$ is also flasque; indeed, this follows from the fact [40, 2.11 and 2.18] that the sequence of pcdh sheaves

$$0 \to \mathbb{Z}_{pcdh}(Z') \to \mathbb{Z}_{pcdh}(Z) \oplus \mathbb{Z}_{pcdh}(X') \to \mathbb{Z}_{pcdh}(X) \to 0$$

is exact for each abstract blowup $X' \to X$ with center Z. (Here, $\mathbb{Z}_{pcdh}(W)$ denotes the pcdh sheafification of $\mathbb{Z}\,\mathrm{Hom}(-, W)$.)

We claim that $F^\bullet \to I^\bullet$ is globally a quasi-isomorphism, so that $F^\bullet(V) \to I^\bullet(V)$ is a quasi-isomorphism for all quasi-projective varieties V. Indeed, by taking cones, it suffices to show that a locally acyclic flasque complex $G^\bullet$ that is bounded below is object-wise acyclic. Observe that the presheaf $h^{n-1}(G^\bullet)(-)$ is zero for n small enough. Let $x \in h^n(G^\bullet)(V)$ for some scheme V. There is an abstract blowup $f : V' \to V$, $Z \subset V$, $Z' = Z \times_V V'$ such that $f^*(x) = 0$ [15, 3.3]. The element x also vanishes on Z by Noetherian induction, and hence lifts to an element of $h^{n-1}(G^\bullet(Z'))$, which is the zero group by the inductive hypothesis.

It remains to prove $I^\bullet(X) \to I^\bullet(X_\bullet)$ is a quasi-isomorphism. But this holds for each fixed I^n since $X_\bullet \to X$ is a hyper-covering for the pcdh topology and I^n is an injective pcdh sheaf. □

Using Proposition 5.2, we have that the bicomplex (5.1) induces a spectral sequence

$$E^2_{p,q} = h_p(\cdots \to H_q(U_1^{an}) \to H_q(U_0^{an})) \Longrightarrow H^{BM}_{p+q}(U^{an}). \tag{5.3}$$

Given an abelian group A, we may also form the evident analogue with coefficients in A

$$E^2_{p,q} = h_p(\cdots \to H_q(U_1^{an}, A) \to H_q(U_0^{an}, A)) \Longrightarrow H^{BM}_{p+q}(U^{an}, A)$$

by first tensoring (5.1) by A over $\mathbb{Z}$. We will refer to this spectral sequence as the Deligne-Gillet-Soulé (DGS for short) spectral sequence.

Of course, the isomorphism type of the singular weight complex of a given variety U depends on the choices made in its construction. However, any two sets of choices made in constructing this bicomplex result in quasi-isomorphic complexes. In fact, even more is true: two sets of choices for the construction of the singular weight complex of a given variety U will result in bicomplexes such that the associated spectral sequences are isomorphic (starting at the E^2-term); indeed, [21, Theorem 2] and the fact that the singular homology functor factors through the category of Chow motives implies that the complexes $\cdots \to H_q(U_1^{an}, A) \to H_q(U_0^{an}, A)$ are well-defined up to canonical homotopy equivalence of complexes. In particular, the following definition (made by Gillet and Soulé for cohomology with compact supports) is independent of all choices made.

Definition 5.4. [21, Theorem 3] For a quasi-projective complex variety U, the *weight filtration* on $H^{BM}_*(U^{an})$ is the increasing filtration given by the filtration on the abutment of the DGS spectral sequence. Explicitly, we define

$$\begin{aligned} W_t H_n^{BM}(U^{an}) \;=\; &\operatorname{image}\big(h_n\big(\mathbb{Z}\operatorname{Sing}(U_{n+t}) \\ &\to \cdots \to \mathbb{Z}\operatorname{Sing}(U_0)\big) \to H_n^{BM}(U^{an})\big), \end{aligned}$$

where the U_i's form the singular weight complex of U. For any abelian group A, we define

$$\begin{aligned} &W_t H_n^{BM}(U^{an}, A) \\ &\quad = \operatorname{image}\big(h_n\big(\mathbb{Z}\operatorname{Sing}(U_{n+t}) \otimes_{\mathbb{Z}} A \\ &\quad \to \cdots \to \mathbb{Z}\operatorname{Sing}(U_0) \otimes_{\mathbb{Z}} A\big) \to H_n^{BM}(U^{an}, A)\big). \end{aligned}$$

The key properties established by Gillet-Soulé for the weight filtration $W_* H_*^{BM}$ are recorded in the following theorem.

Theorem 5.5 (Gillet-Soulé). *Let U be a quasi-projective complex variety and A an abelian group.*

(1) *The filtration* $W_* H_*^{BM}(U^{an}, A)$ *is independent of all the choices made.*
(2) *If* U *has dimension* d*, then the weight filtration on* $H_n^{BM}(U^{an}, A)$ *is supported in the range* $[-n, d-n]$ *— that is,* $W_t H_n^{BM}(U^{an}, A) = 0$ *for* $t < -n$ *and* $W_t H_n^{BM}(U^{an}, A) = H_n^{BM}(U^{an}, A)$ *for* $t \geq d - n$.
(3) *If* U *is smooth and projective, then* $H_n^{BM}(U^{an}, A) = H_n(U^{an}, A)$ *is of pure weight* $-n$ *— that is,* $W_t H_n(U^{an}, A)$ *vanishes for* $t < -n$ *and equals* $H_n(U^{an}, A)$ *for* $t \geq -n$.
(4) *The weight filtration is functorial for closed immersions: if* $j : Z \subset U$ *is a closed immersion, then the canonical map* $j_* : H_*^{BM}(Z^{an}, A) \to H_*^{BM}(U^{an}, A)$ *restricts to a map* $j_* : W_t H_*^{BM}(Z^{an}, A) \to W_t H_*^{BM}(U^{an}, A)$ *for all* t.
(5) *The weight filtration is functorial for open immersions : if* $i : V \rightarrowtail U$ *is an open immersion, then the canonical map* $i^* : H_*^{BM}(U^{an}, A) \to H_*^{BM}(V^{an}, A)$ *restricts to a map* $i^* : W_t H_*^{BM}(U^{an}, A) \to W_t H_*^{BM}(V^{an}, A)$ *for all* t.
(6) *The weight filtration is compatible with localization sequences: if* $j : V \rightarrowtail U$ *is the open complement of a closed immersion* $j : Z \to U$*, then the boundary map in the localization sequence for Borel-Moore homology* $H_*^{BM}(V^{an}, A) \to H_{*-1}^{BM}(Z^{an}, A)$ *restricts to a map* $W_t H_*^{BM}(V^{an}, A) \to W_t H_{*-1}^{BM}(Z^{an}, A)$*, for all* t.

Remark 5.6. We have *not* asserted that localization sequences for Borel-Moore homology restrict to give long exact sequences involving the functors $W_t H_*^{BM}$, for a fixed t. Indeed, this is not the case in general, and one motivation for introducing the groups $\tilde{W}_t H_*^{BM}$ below is to rectify this behavior.

As with any (convergent) spectral sequence, we can describe the filtration on the abutment using the D^r-terms for any r instead of the D^1-terms as in Definition 5.4. Specifically, we use the D^2-terms which we identify using the following fact from homological algebra :

Lemma 5.7. *Let* A *be a (bounded above) double (chain) complex. Let* $tr^v_{\geq j} A$ *denote the double complex obtained by "good" truncation in the vertical direction and* $t^h_{\leq i} A$ *denote the double complex obtained by "brutal" truncation in the horizontal direction. Write* $F^v_\bullet Tot(A)$ *and* $F^h_\bullet Tot(A)$ *for the the corresponding filtrations on the total complex associated to* A*, and let* $E(v)$ *and* $E(h)$ *refer to the corresponding spectral sequences. Then* $E^1(v)$ *and* $E^2(h)$ *are isomorphic as differential graded modules — that is, the two spectral sequences coincide up to reindexing.*

Using this, we can now give an alternative description of the weight filtration as follows :

$$\begin{aligned} W_t H_n^{BM}(U^{an}) = \text{ image} \big[h_n(\cdots \to tr_{\geq -t} \mathbb{Z}Sing(U_1) \\ \to tr_{\geq -t} \mathbb{Z}Sing(U_0)) \to H_n^{BM}(U^{an}) \big], \end{aligned}$$

where in general $tr_{\geq j}$ denotes the "good truncation" of chain complexes at degree j. Motivated by this alternative description of $W_* H_*^{BM}$, we introduce the following groups:

Definition 5.8. Given a quasi-projective complex variety U, define

$$\tilde{W}_t H_n^{BM}(U) = h_n(\cdots \to tr_{\geq -t}\mathbb{Z}\,\mathrm{Sing}(U_1) \to tr_{\geq -t}\mathbb{Z}\,\mathrm{Sing}(U_0)).$$

For any abelian group A, define

$$\begin{aligned}\tilde{W}_t H_n^{BM}(U, A) \;=\; & h_n(\cdots \to tr_{\geq -t}(\mathbb{Z}\,\mathrm{Sing}(U_1) \otimes_{\mathbb{Z}} A) \\ & \to tr_{\geq -t}(\mathbb{Z}\,\mathrm{Sing}(U_0) \otimes_{\mathbb{Z}} A)).\end{aligned}$$

In other words, the group $\tilde{W}_t H_n(U)$ is the $D^2_{n+t,-t}$-term of the DGS spectral sequence. In particular, there is a canonical surjective map

$$\tilde{W}_t H_n^{BM}(U) \to W_t H_n^{BM}(U^{an})$$

given by mapping D_2-terms to D_∞-terms. In certain cases, this map is an isomorphism, but, in general, the $\tilde{W}_t H_*^{BM}$ are better behaved than the weight filtration itself.

The following result implies that the groups $\tilde{W}_t H_n^{BM}$ are independent of the choices made up to canonical isomorphism; it is a restatement in our context of [21, Theorem 2], which asserts the existence and well-definedness of the "motivic weight complex" functor.

Proposition 5.9. *Suppose we have chosen a projective compactification $U \subset X$ for every quasi-projective variety U and a smooth hyper-envelope $Y_\bullet \to X_\bullet$ of $Y \rightarrowtail X$ for every pair (X, Y) consisting of a projective variety X and a closed subvariety Y. Fix an integer t and an abelian group A. Then these choices determine a functor*

$$WC_t(-, A) : \mathbf{Var}^{prop}/\mathbb{C} \to \mathcal{D}_+(\mathbf{Ab})$$

from the category of complex quasi-projective varieties and proper morphisms to the derived category of bounded below chain complexes of abelian groups, defined by

$$WC_t(U, A) = Tot(\cdots \to tr_{\geq -t}(\mathbb{Z}\,\mathrm{Sing}(U_1)\otimes_{\mathbb{Z}} A) \to tr_{\geq -t}(\mathbb{Z}\,\mathrm{Sing}(U_0)\otimes_{\mathbb{Z}} A)).$$

Moreover, the assignment $U \mapsto WC_t(U, A)$ is contravariantly functorial for open immersions. Finally, a different set of choices of compactifications and smooth hyper-envelopes determines a canonically isomorphic functor.

Proof. This is proved exactly as in [21, 2.2 and 2.3]. Observe that all maps given there for the definition of the functor (in [21], to the homotopy category of complexes of Chow motives $\mathbf{Hot}(\mathbf{M})$) are actually either induced by morphisms of simplicial varieties or are inverses (in $\mathbf{Hot}(\mathbf{M})$) of maps induced by morphisms of simplicial varieties. Observe that any morphism of simplicial varieties induces a map of their truncated singular complexes, and those morphisms that become invertible in the category **Hot**(**M**) induce quasi-isomorphisms on truncated singular complexes, since the homology functors H_n on smooth projective varieties factor through the category of pure effective Chow motives. Thus, the argument goes through without difficulty. □

Remark 5.10. Observe that we have $\tilde{W}_t H_n^{BM} = h_n \circ WC_t$, by definition.

Theorem 5.11. *The functors $\tilde{W}_* H_*^{BM}$ enjoy the following properties for quasi-projective complex varieties:*

(1) *The groups $\tilde{W}_* H_*^{BM}$ are independent of the choices made, up to canonical isomorphism. Moreover, (having fixed a set of choices as in Proposition 5.9) there is a natural transformation*

$$\tilde{W}_t H_n^{BM} \to \tilde{W}_{t+1} H_n^{BM}, \quad \text{for each } t, n \in \mathbb{Z},$$

that is an isomorphism if $t \geq 0$, and there is a natural isomorphism $\tilde{W}_0 H_n^{BM} \cong H_n^{BM}$, for each n.

(2) *For any quasi-projective complex variety U and abelian group A, there is a surjective map*

$$\tilde{W}_t H_n^{BM}(U, A) \to W_t H_n^{BM}(U^{an}, A)$$

which is covariantly functorial for proper morphisms and contravariantly functorial for open inclusions.

(3) *The map $\tilde{W}_t H_n(U, A) \to W_t H_n(U^{an}, A)$ is an isomorphism for all t, n if and only if the DGS spectral sequence with coefficients in A degenerates at E_2. In particular, this map is an isomorphism (for any coefficients) if U is a smooth, projective variety or U is the open complement of a closed immersion of smooth, projective varieties. Moreover, this map is isomorphism for any U provided $A = \mathbb{Q}$.*

(4) *Given an open immersion $V \subset U$ with closed complement Z there are functorial long exact localization sequences*

$$\cdots \to \tilde{W}_t H_n^{BM}(Z, A) \to \tilde{W}_t H_n^{BM}(U, A) \to \tilde{W}_t H_n^{BM}(V, A) \to \tilde{W}_t H_{n-1}^{BM}(Z, A) \to \cdots$$

for all t whose maps commute with the natural transformations $\tilde{W}_t H_*^{BM} \to \tilde{W}_{t+1} H_*^{BM}$ and in particular with the maps of the long exact localization sequence for Borel-Moore homology

$$\cdots \to H_n^{BM}(Z^{an}, A) \to H_n^{BM}(U^{an}, A) \to H_n^{BM}(V^{an}, A) \to H_{n-1}^{BM}(Z^{an}, A) \to \cdots.$$

Proof. We are justified in using the word "functor" by Proposition 5.9, which also implies the independence of choices up to canonical isomorphism. The natural transformations are the obvious ones and the fact that $\tilde{W}_0 H^{BM} \cong H^{BM}$ is the content of Proposition 5.2. This shows (1).

The existence and surjectivity of the map in (2) follows immediately from the observation that $\tilde{W}_t H_n^{BM}$ and $W_t H_n^{BM}$ are the D^2- and D^∞-terms of the DGS spectral sequence.

The first assertion of (3) is just the general fact that a spectral sequence degenerates at E^2 if and only if the natural map $D^2 \to D^\infty$ is an isomorphism. If X and $Z \subset X$ are smooth and projective, then we can choose them to be their own smooth hyper-envelopes. This implies that in the DGS spectral sequence $E^2_{p,q}$ vanishes for $p \neq 0, 1$ and thus all differentials vanish. The fact that the spectral sequence with rational coefficients degenerates for any U has been shown by Deligne [8] for cohomology with compact supports; but with rational coefficients, cohomology and homology are dual to one another by the universal coefficient theorem, and thus so are these spectral sequences.

Assertion (4) is proven in the same way as [21, Theorem 2, (iii)]. In detail, choose a compactification X of U and compactify Z by $Y = X - V$. The result is a square

$$\begin{array}{ccc} Y - Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X - U & \longrightarrow & X \end{array}$$

of closed embeddings, which can be covered by a square of simplicial smooth projective varieties such that the maps on vertices are smooth hyper-envelopes. Applying any of the functors WC_t gives a diagram of objects in $\mathcal{D}_+(\mathbf{Ab})$ of the form

$$\begin{array}{ccccc} WC_t(Y - Z) & \longrightarrow & WC_t(Y) & \longrightarrow & WC_t(Z) \\ \downarrow & & \downarrow & & \downarrow \\ WC_t(X - U) & \longrightarrow & WC_t(X) & \longrightarrow & WC_t(U) \\ \downarrow & & \downarrow & & \downarrow \\ WC_t(X - (U \cup Y)) & \longrightarrow & WC_t(X - Y) & \longrightarrow & WC_t(V) \end{array}$$

in which the first two columns and the first two rows are distinguished triangles (basically, by definition) and the bottom row is a distinguished triangle since $X - (U \cup Y)$ is empty and $X - Y = V$; hence, so is the rightmost column. □

Theorem 5.12. *For any quasi-projective complex variety U, the map from Lawson homology to Borel-Moore homology factors as*

$$L_t H_n(U) \to \tilde{W}_{-2t} H_n^{BM}(U) \to H_n^{BM}(U^{an}), \quad \textit{for all } n, t \in \mathbb{Z}.$$

Moreover, for $t \geq 0$ the first map is given by a map in the derived category of abelian groups

$$\operatorname{Sing} Z_t(U)[-2t] \to WC_{-2t}(U)$$

that is covariantly natural for proper morphisms and contravariantly natural for open immersions. Thus the map $L_tH_*(-) \to \tilde{W}_{-2t}H_*(-)$ *is compatible with localization long exact sequences.*

Proof. Since $L_tH_n = L_0H_n$ and $\tilde{W}_{-2t}H_n^{BM} = \tilde{W}_0H_n^{BM} = H_n^{BM}$ for $t \leq 0$, we may assume $t \geq 0$. We begin by establishing a chain of natural transformations joining the functors $\mathrm{Sing}\, Z_t(-)[-2t]$ and $tr_{\geq 2t}\mathbb{Z}\,\mathrm{Sing}(-)$. Let $s \in \mathrm{Sing}_{2t}\, Z_0(\mathbb{A}^t)$ be a representative for a generator of $h_{2t}(\mathrm{Sing}\, Z_0(\mathbb{A}^t)) \cong \mathbb{Z}$ and let

$$s : \mathrm{Sing}\, Z_t(U)[-2t] \to \mathrm{Sing}\, Z_t(U \times \mathbb{A}^t)$$

be the map induced by external product with the element s. Clearly this map factors uniquely through a map of the form

$$s : \mathrm{Sing}\, Z_t(U)[-2t] \to tr_{\geq 2t}\,\mathrm{Sing}\, Z_t(U \times \mathbb{A}^t).$$

By [10], the natural map

$$\mathrm{Sing}\, Z_0(U) \to \mathrm{Sing}\, Z_t(U \times \mathbb{A}^t)$$

given by flat pullback along $U \times \mathbb{A}^t \to U$ is a quasi-isomorphism. Observe that $Z_0(U)$ may be identified with $\coprod_n Symm^n(U^{an})^+$, the group completion of the infinite symmetric product of the space U^{an} and thus we have a map

$$\mathbb{Z}\,\mathrm{Sing}(U) \to \mathrm{Sing}\, Z_0(U)$$

which is a quasi-isomorphism by the Dold-Thom Theorem. We therefore have a natural quasi-isomorphism of the form

$$\mathbb{Z}\,\mathrm{Sing}(U) \xrightarrow{\sim} \mathrm{Sing}\, Z_t(U \times \mathbb{A}^t).$$

The chain of natural transformations we seek is given by the diagram

$$\mathrm{Sing}\, Z_t(U)[-2t] \xrightarrow{\ s\ } tr_{\geq 2t}\,\mathrm{Sing}\, Z_t(U \times \mathbb{A}^t) \xleftarrow{\ \sim\ } tr_{\geq 2t}\mathbb{Z}\,\mathrm{Sing}(U).$$

As usual, we choose a compactification X of U with reduced complement Y and a smooth hyper-envelope $Y_\bullet \to X_\bullet$ of the closed immersion $Y \rightarrowtail X$. Let $U_i = X_i \coprod Y_{i-1}$ and apply each of the functors $\mathrm{Sing}\, Z_t(-)$, $\mathrm{Sing}\, Z_t(- \times \mathbb{A}^t)$, and $\mathbb{Z}\,\mathrm{Sing}(-)$ to each U_i. Taking appropriate alternating sums of maps, we obtain a chain of maps of bicomplexes

$$\mathrm{Sing}\, Z_t(U_\bullet)[-2t] \xrightarrow{\ s\ } \mathrm{Sing}\, tr_{\geq 2t} Z_t(U_\bullet \times \mathbb{A}^t) \xleftarrow{\ \sim\ } tr_{\geq 2t}\mathbb{Z}\,\mathrm{Sing}(U_\bullet).$$

Taking homology groups of the associated total complexes and using Proposition 5.2, we obtain the desired map

$$L_tH_n(U) \to \tilde{W}_{-2t}H_n^{BM}(U^{an}).$$

It is apparent from the construction that the composition of this map with the map $\tilde{W}_{-2t}H_n^{BM}(U^{an}) \to H_n^{BM}(U^{an})$ gives the usual map from Lawson homology to Borel-Moore homology. □

Corollary 5.13. *For any quasi-projective complex variety U, the image of the canonical map $L_t H_n(U) \to H_n^{BM}(U^{an})$ lies in the part of weight at most $-2t$ of Borel-Moore homology.*

We close this section by establishing a suitable version of the projective bundle formula for the functors $\tilde{W}_* H_n^{BM}$. Recall that for a chain complex of abelian groups A, we write $A[i]$ for the chain complex defined by $A[i]_r = A_{r+i}$.

Theorem 5.14. *Let X be a quasi-projective variety and $E \to X$ a vector bundle of rank $c+1$. For any integer t and abelian group A, there is an isomorphism in the derived category of the form*

$$\bigoplus_{i=0}^{c} WC_{t+2i}(X, A)[-2i] \cong WC_t(\mathbb{P}(E), A), \tag{5.15}$$

and hence there is an isomorphism

$$\bigoplus_{i=0}^{c} \tilde{W}_{t+2i} H_{n-2i}^{BM}(X, A) \cong \tilde{W}_t H_n^{BM}(\mathbb{P}(E), A). \tag{5.16}$$

Moreover, the isomorphism (5.16) can be chosen to be compatible with the projective bundle formulas in Lawson [10] and Borel-Moore homology under the maps of Theorem 5.12.

Remark 5.17. We have not asserted that there exist isomorphisms as in Theorem 5.14 that are natural with respect to pullback along open immersion or pushforward along proper morphisms. Presumably such natural isomorphisms exist, but establishing their existence appears to be a delicate matter.

In order to prove the theorem, we need some auxiliary results and constructions which might be of independent interest. First, we need the following analogue of Proposition 5.9.

Proposition 5.18. *Suppose we have chosen projective compactifications and smooth hyper-envelopes of closed immersions of smooth projective varieties as in Proposition 5.9. For any integer t, natural number r and abelian group A, these choices determine a functor*

$$WC_{t,r}(-, A) : \mathbf{Var}^{prop}/\mathbb{C} \to \mathcal{D}_+(\mathbf{Ab})$$

from the category of complex quasi-projective varieties and proper morphisms to the derived category of bounded below chain complexes of abelian groups, defined by

$$\begin{aligned} WC_{t,r}(U, A) \;=\; & Tot(\cdots \to tr_{\geq -t}(\operatorname{Sing} Z_r(U_1) \otimes_{\mathbb{Z}} A) \\ & \to tr_{\geq -t}(\operatorname{Sing} Z_r(U_0) \otimes_{\mathbb{Z}} A)). \end{aligned}$$

Moreover, the assignment $U \mapsto WC_{t,r}(U, A)$ is contravariantly functorial for open immersions. Finally, a different set of choices of compactifications and smooth hyper-envelopes determines a canonically isomorphic functor.

Proof. By [12, 7.2] the Lawson homology functors $L_r H_*$ on smooth projective varieties are functorial with respect to equidimensional correspondences (via "correspondence homomorphisms"). Since any d-dimensional cycle on a product $X \times Y$ of smooth projective varieties with $\dim(X) = d$ is rationally equivalent to a cycle finite and surjective over X, we conclude that the Lawson homology functors $L_r H_*$ on smooth projective varieties factor through the category of pure, effective Chow motives. The argument of the proof of Proposition 5.9 thus applies without change to complete the proof. □

Next, we collect some properties of the functors $WC_{t,r}$, dropping the coefficients from the notation.

Proposition 5.19. *We have the following natural transformations and natural isomorphisms (in the derived category) involving the functors $WC_{t,r}$.*

(1) *For each t, r, there is a natural transformation $WC_{t,r} \to WC_{t+1,r}$ which is a natural isomorphism for $t \geq 0$.*
(2) *There is a natural isomorphism of the form $WC_t \xrightarrow{\cong} WC_{t,0}$ for each $t \in \mathbb{Z}$.*
(3) *There is a natural isomorphism $WC_{0,r} \xrightarrow{\cong} \mathrm{Sing}\, Z_r$ for each $r \geq 0$.*
(4) *There are natural transformations $s : WC_{t,r} \to WC_{t-2,r-1}[2]$. When $t \geq 2$, these maps coincide with the usual s-map in Lawson homology under the isomorphisms of (1) and (3). In general, these maps commute with the natural transformation of (1).*

Proof. Let F be a functor defined on the category of smooth, projective complex varieties with values in chain complexes of abelian groups such that the functors $h_n(F)$ on smooth projective varieties factor through pure effective Chow motives and let $\mathcal{F}$ denote the induced functor from $\mathbf{Var}^{prop}/\mathbb{C}$ to $\mathcal{D}_+(\mathbf{Ab})$ defined by applying F degree-wise to smooth hyper-envelopes and then taking total complexes. (Here, we have implicitly chosen projective compactifications and smooth hyper-envelopes as in Proposition 5.9.) Thus, if $F = tr_{\geq -t}\, \mathrm{Sing}\, Z_r(-)$, then $\mathcal{F} = WC_{t,r}$; if $F = tr_{\geq -t}\mathbb{Z}\, \mathrm{Sing}(-)$, then $\mathcal{F} = WC_t$.

Clearly, a natural transformation of such functors $F \to F'$ induces a natural transformation of induced functors $\mathcal{F} \to \mathcal{F}'$ from $\mathbf{Var}^{prop}/\mathbb{C}$ to $\mathcal{D}_+(\mathbf{Ab})$. In particular, the natural transformations of (1) and (2) are induced by the natural transformations $tr_{\geq -t}\, \mathrm{Sing}\, Z_r(-) \to tr_{\geq -t-1}\, \mathrm{Sing}\, Z_r(-)$ and $tr_{\geq -t}\mathbb{Z}\, \mathrm{Sing}(-) \to tr_{\geq -t}\, \mathrm{Sing}\, Z_0(-)$ in this manner.

To obtain (4), we choose a representative of a generator of $h_2(\mathrm{Sing}\, Z_0(\mathbb{A}^1))$ (as in the proof of Theorem 5.12) and use exterior product and flat pullback to define natural transformations

$$\mathrm{Sing}\, Z_r(-) \longrightarrow \mathrm{Sing}\, Z_r(- \times \mathbb{A}^1)[2] \longleftarrow \mathrm{Sing}\, Z_{r-1}(-)[2].$$

Applying $tr_{\geq -t}$ to this chain gives a pair of natural transformations that induce natural transformations

$$WC_{t,r} \longrightarrow \mathcal{F} \longleftarrow WC_{t-2,r-1}[2],$$

where $\mathcal{F}$ is induced by $F = tr_{\geq -t}\operatorname{Sing} Z_r(- \times \mathbb{A}^1)[2]$ as above. Since $\operatorname{Sing} Z_{r-1}(-)[2] \to \operatorname{Sing} Z_r(- \times \mathbb{A}^1)[2]$ is a natural isomorphism, so is $WC_{t-2,r-1}[2] \to \mathcal{F}$, and we define s to be the evident composition. The asserted compatibility results are evident from the construction.

Finally, assertion (3) is the content of Proposition 5.2. □

Proof of Theorem 5.14. We first show that we can find a compactification $\overline{X}$ of X and a vector bundle $E_{\overline{X}} \to \overline{X}$ whose restriction to X is $E \to X$. To see this, let $E \to X$ correspond to the locally free coherent sheaf $\mathcal{F}$ on X. Now choose any compactification X' of X and extend $\mathcal{F}$ to a coherent sheaf $\mathcal{F}'$ on X'. By the platification par éclatement theorem [31, 5.2], there is a blow-up $\overline{X}$ of X' away from X such that the proper transform $\overline{\mathcal{F}}$ of $\mathcal{F}'$ is flat and hence locally free. The locally free sheaf $\overline{\mathcal{F}}$ corresponds to a vector bundle $E_{\overline{X}} \to \overline{X}$ whose restriction to X is $E \to X$.

Let $Y = \overline{X} - X$ and let $E_Y \to Y$ denote the restriction of $E_{\overline{X}} \to \overline{X}$ to Y. Choose a smooth hyper-envelope $Y_\bullet \to X_\bullet$ of $Y \rightarrowtail X$, and let $E_{Y_\bullet}$ and $E_{\overline{X}_\bullet}$ denote the evident pullbacks. Taking associated projectivized bundles, we obtain a commutative square of simplicial varieties

$$\begin{array}{ccc} \mathbb{P}(E_{Y_\bullet}) & \longrightarrow & \mathbb{P}(E_{\overline{X}_\bullet}) \\ \downarrow & & \downarrow \\ Y_\bullet & \longrightarrow & \overline{X}_\bullet \end{array} \tag{5.20}$$

and an augmentation map from this square to the commutative square

$$\begin{array}{ccc} \mathbb{P}(E_Y) & \longrightarrow & \mathbb{P}(E_{\overline{X}}) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \overline{X}. \end{array}$$

One may readily verify that $\mathbb{P}(E_{Y_\bullet}) \to \mathbb{P}(E_{\overline{X}_\bullet})$ is a smooth hyper-envelope of $\mathbb{P}(E_Y) \rightarrowtail \mathbb{P}(E_{\overline{X}})$, and hence $WC_{t,r}(\mathbb{P}(E), A)$ is isomorphic to the complex obtained by applying $tr_{\geq -t}\operatorname{Sing} Z_r(-) \otimes_{\mathbb{Z}} A$ degree-wise to each of the varieties comprising $\mathbb{P}(E_{Y_\bullet})$ and $\mathbb{P}(E_{\overline{X}_\bullet})$ and taking total complexes. In a similar fashion, the map $Y_\bullet \to \overline{X}_\bullet$ can be used to define $WC_{t,r}(X, A)$. Moreover, note that the vertical maps of (5.20) are flat of relative dimension c, and so we obtain the commutative square

$$\begin{array}{ccc} Z_r(Y_\bullet) & \longrightarrow & Z_{r+c}(\mathbb{P}(E_{Y_\bullet})) \\ \downarrow & & \downarrow \\ Z_r(\overline{X}_\bullet) & \longrightarrow & Z_{r+c}(\mathbb{P}(E_{\overline{X}_\bullet})). \end{array} \tag{5.21}$$

If W is a smooth, projective variety and $L \to W$ is a line bundle, then we have natural maps

$$Z_r(W) \longrightarrow Z_r(L) \xleftarrow{\sim} Z_{r-1}(W),$$

for all r, given by pushforward along the zero section $W \rightarrowtail L$ and flat pullback along $L \to W$. The composition of these maps (in the derived category) gives the "Chern class operator" induced by the line bundle L on the Lawson homology of W. Letting W range over the varieties comprising the simplicial varieties $\mathbb{P}(E_{Y_\bullet})$ and $\mathbb{P}(E_{\overline{X}_\bullet})$ and letting L range over the canonical line bundles, we obtain for each $i \geq 0$ a commutative diagram of the form

$$\begin{array}{ccccc} Z_{r+c-i}(\mathbb{P}(E_{Y_\bullet})) & \longrightarrow & Z_{r+c-i}(L_{Y_\bullet}) & \xleftarrow{\sim} & Z_{r+c-i-1}(\mathbb{P}(E_{Y_\bullet})) \\ \downarrow & & \downarrow & & \downarrow \\ Z_{r+c-i}(\mathbb{P}(E_{\overline{X}_\bullet})) & \longrightarrow & Z_{r+c-i}(L_{\overline{X}_\bullet}) & \xleftarrow{\sim} & Z_{r+c-i-1}(\mathbb{P}(E_{\overline{X}_\bullet})) \end{array} \tag{5.22}$$

in which all vertical maps are given by pushforward along proper morphisms. (Observe that each canonical line bundle is obtained by pullback from the canonical line bundle $L_{\overline{X}_0}$.)

Applying $tr_{\geq -t}\operatorname{Sing}(-)$ degree-wise to the diagrams (5.21) and (5.22) gives us maps of the form

$$WC_{t,r}(X) \to WC_{t,r+c}(\mathbb{P}(E))$$

and

$$WC_{t,r+c-i}(\mathbb{P}(E)) \to WC_{t,r+c-i-1}(\mathbb{P}(E)), \quad \text{for all } r+c-1 \geq i \geq 0.$$

Taking compositions in the evident manner gives us maps of the form

$$WC_{t,r}(X) \to WC_{t,r+c-i}(\mathbb{P}(E)), \quad \text{for all } c \geq i \geq 0,$$

which upon composition with s^{c-i} (where s is the map of Proposition 5.19) yields the map

$$WC_{t,r}(X) \to WC_{t-2c+2i,r}(\mathbb{P}(E))[2c-2i].$$

By adding these maps together and reindexing we get the map

$$\bigoplus_{i=0}^{c} WC_{t+2i,r}(X)[-2i] \to WC_{t,r}(\mathbb{P}(E)). \tag{5.23}$$

When $r = 0$, this is the map we seek (since $WC_{*,0} \cong WC_*$ by part (2) of Proposition 5.19).

To prove (5.23) is an isomorphism in the derived category for $r = 0$, observe that it follows from its construction that this map fits into the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{i=0}^{c} WC_{t+2i,0}(Y)[-2i] & \longrightarrow & \bigoplus_{i=0}^{c} WC_{t+2i,0}(\overline{X})[-2i] & \longrightarrow & \bigoplus_{i=0}^{c} WC_{t+2i,0}(X)[-2i] \\ \downarrow & & \downarrow & & \downarrow \\ WC_{t,0}(\mathbb{P}(E_Y)) & \longrightarrow & WC_{t,0}(\mathbb{P}(E_{\overline{X}})) & \longrightarrow & WC_{t,0}(\mathbb{P}(E)) \end{array} \tag{5.24}$$

whose rows are distinguished triangles, and so it suffices to prove the left-hand and middle vertical arrows of this diagram are isomorphisms in the derived category. But these maps are obtained from the collection of maps of the form

$$\bigoplus_{i=0}^{c} \operatorname{Sing} Z_0(W)[-2i] \to \operatorname{Sing} Z_0(\mathbb{P}(E_W)), \tag{5.25}$$

for $W = Y_n$ or $\overline{X}_m$, given by the composition of flat pullback, the Chern class operators, and the s-map:

$$\begin{array}{ccc} \bigoplus_{i=0}^{c} \operatorname{Sing} Z_0(W)[-2i] & \longrightarrow & \operatorname{Sing} Z_0(\mathbb{P}(E_W)) \\ \downarrow & & \uparrow \\ \bigoplus_{i=0}^{c} \operatorname{Sing} Z_c(\mathbb{P}(E_W))[-2i] & \longrightarrow & \bigoplus_{i=0}^{c} \operatorname{Sing} Z_i(\mathbb{P}(E_W))[-2i]. \end{array}$$

We claim (5.25) is a quasi-isomorphism for any quasi-projective variety W. Indeed, since the s map is readily verified to commute with flat pullback and proper pushforward, the map (5.25) fits into the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=0}^{c} Z_0(W)[-2i] & \longrightarrow & Z_0(\mathbb{P}(E_W)) \\ \sim\downarrow & & \sim\downarrow \\ \bigoplus_{i=0}^{c} Z_c(W \times \mathbb{A}^c)[-2i] & \longrightarrow & Z_c(\mathbb{P}(E_{W\times\mathbb{A}^c})) \\ \oplus s^i\downarrow & & =\downarrow \\ \bigoplus_{i=0}^{c} Z_{c-i}(W \times \mathbb{A}^c) & \longrightarrow & Z_c(\mathbb{P}(E_{W\times\mathbb{A}^c})), \end{array}$$

whose bottom map is the quasi-isomorphism that gives the projective bundle formula for Lawson homology. The vertical maps in this diagram are also quasi-isomorphisms, and hence the map (5.25) is a quasi-isomorphism as claimed. It remains a quasi-isomorphism upon applying $tr_{\geq -t}$, and the left-hand and middle vertical maps of (5.24) are given by taking total complexes of these quasi-isomorphisms. Hence they are isomorphisms (in the derived category).

Finally, to prove the compatibility of the isomorphism (5.16) with the bundle formulas for Lawson and Borel-Moore homology under the maps of Theorem 5.12, observe that each of the maps used to construct (5.23) commutes with the maps given in (1) and (4) of Proposition 5.19. Hence the map (5.23) commutes with these transformations, and thus we have that

$$\begin{array}{ccc} \bigoplus_{i=0}^{c} WC_{t+2i,r}(X)[-2i] & \longrightarrow & WC_{t,r}(\mathbb{P}(E)) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ \bigoplus_{i=0}^{c} WC_{t+2i-2,r-1}(X)[-2i+2] & \longrightarrow & WC_{t-2,r-1}(\mathbb{P}(E))[2] \end{array}$$

and

$$\begin{array}{ccc} \bigoplus_{i=0}^{c} WC_{t+2i,r}(X)[-2i] & \longrightarrow & WC_{t,r}(\mathbb{P}(E)) \\ \downarrow & & \downarrow \\ \bigoplus_{i=0}^{c} WC_{t+1+2i,r}(X)[-2i] & \longrightarrow & WC_{t+1,r}(\mathbb{P}(E)) \end{array}$$

commute. This suffices to prove the compatibility of bundle formulas, since $WC_{t,r}$ gives Lawson homology and $WC_{t,0}$ gives Borel-Moore homology whenever $t \geq 0$. □

6. Comparing homology, cohomology, and K-theory

In this section, we give some conditions on the morphic cohomology of a variety that ensure an isomorphism of the semi-topological and topological K-theory of the variety in a certain range (see Theorem 6.1). We then define a class $\mathcal{C}$ of varieties for which the refined cycle maps of Theorem 5.12 are isomorphisms, verifying that a variety that is both smooth and in $\mathcal{C}$ satisfies the conditions of Theorem 6.1. We provide a number of examples of varieties in $\mathcal{C}$.

Theorem 6.1. *Let X be a smooth quasi-projective complex variety X and let A be an abelian group.*

(1) *If*

$$L^q H^n(X, A) \to H^n(X^{an}, A)$$

is an isomorphism for $n \leq 2q$ then the map

$$K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$$

is an isomorphism for $i \geq 0$.

(2) *If*

$$L^q H^n(X, A) \to H^n(X^{an}, A)$$

is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$, then the map

$$K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$$

is an isomorphism for $i \geq \dim(X) - 1$ and a monomorphism for $i = \dim(X) - 2$.

Proof. This follows from Theorem 3.6 in the same way as Theorem 3.7 (which is just a special case of this theorem). We only give a proof for the second statement, which is slightly more complicated than the first. Let d denote the dimension of X and let $E(sst)$ (resp., $E(top)$) denote the semi-topological (resp., topological) spectral sequence. Under the given assumptions, we prove the following about the maps $E_r(sst) \to E_r(top)$, for all $r \geq 2$:

(1) The map $E_r^{p,q}(sst) \to E_r^{p,q}(top)$ is an isomorphism provided $p+q \leq 1-d$.
(2) The map $E_r^{p,q}(sst) \to E_r^{p,q}(top)$ is a monomorphism provided $p+q \leq 2-d$

Observe that if $p+q \leq 1-d$ then either $p \leq 0$ or $q \leq -d$, and if $p+q = 2-d$, then either $p \leq 1$ or $q \leq -d$. When $r = 2$, assertion (1) holds for $p \leq 0$ and assertion (2) holds for $p \leq 1$ by hypothesis, and both assertions hold for $q \leq -d$ by duality. We proceed by induction on r using the commutative diagram

$$\begin{array}{ccccc} E_r^{p-r,q+r-1}(sst) & \xrightarrow{d_r^{p-r,q+r-1}} & E_r^{p,q}(sst) & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1}(sst) \\ \downarrow & & \downarrow & & \downarrow \\ E_r^{p-r,q+r-1}(top) & \xrightarrow{d_r^{p-r,q+r-1}} & E_r^{p,q}(top) & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1}(top), \end{array}$$

noting that $E_{r+1}^{p,q}(sst)$ and $E_{r+1}^{p,q}(sst)$ are defined as the middle homology groups of the rows. An easy diagram chase shows that assertions (1) and (2) hold for $r+1$ provided they hold for r.

Consequently, assertions (1) and (2) hold for E_∞-terms as well, and thus the map $K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$ admits a finite filtration whose quotients are isomorphisms for $i \geq d-1$ and monomorphisms for $i = d-2$. The result follows. □

To apply the preceding theorem, we need examples for which the hypotheses hold. These will be furnished by the methods of the previous section; we therefore translate the conditions of the theorem to homology.

Corollary 6.2. *Let A be an abelian group. Assume X is a smooth, quasi-projective complex variety of dimension d such that the generalized cycle map*

$$L_t H_n(X, A) \longrightarrow H_n^{BM}(X^{an}, A)$$

is an isomorphism for $n \geq d+t$ and a monomorphism for $n = d+t-1$ (resp., an isomorphism for $n \geq 2t$). Then the map in K-theory

$$K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$$

is an isomorphism for $i \geq d-1$ and a monomorphism for $i = d-2$ (resp., an isomorphism for $i \geq 0$).

Proof. This is a simple restatement of the theorem, using duality [11]. □

Theorem 6.3. *Assume X is a smooth quasi-projective variety of dimension d such that the refined cycle map (Theorem 5.12)*

$$L_t H_n(X) \to \tilde{W}_{-2t} H_n^{BM}(X)$$

is an isomorphism for all $t, n \in \mathbb{Z}$. Then for any abelian group A, the canonical map

$$K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$$

is an isomorphism for all $i \geq d - 1$ and a monomorphism for $i = d - 2$. If X is projective, this map is an isomorphism for all $i \geq 0$.

Proof. We have to prove that the conditions of Corollary 6.2 are satisfied. First of all, if the refined cycle map is an isomorphism for all t and n, then the same is true with coefficients in any abelian group A. If X happens to be projective, then the refined weight groups coincide with the usual ones (part (3) of Theorem 5.11) and the weights are pure (part (3) of Theorem 5.5) so that the generalized cycle map $L_t H_n(X) \to H_n^{BM}(X^{an})$ is an isomorphism for $n \geq 2t$, as needed.

In general, it suffices to prove that each map in the sequence

$$\tilde{W}_{-2t} H_n^{BM}(X, A) \to \tilde{W}_{-2t+1} H_n^{BM}(X, A) \to \cdots \to \tilde{W}_0 H_n^{BM}(X, A) = H_n^{BM}(X^{an}, A) \qquad (6.4)$$

is an isomorphism if $n \geq d + t$ and is a monomorphism if $n = d + t - 1$. Recall that $\tilde{W}_{-2t+j} H_n^{BM}(X, A)$ is the $D^2_{n-2t+j,2t-j}$-term of the DGS spectral sequence and that the above maps are the usual maps between D^2-terms in an exact couple, and so we have the exact sequence

$$E^2_{n-2t+j+2,2t-j-1} \to \tilde{W}_{-2t+j} H_n^{BM}(X, A) \to \tilde{W}_{-2t+j+1} H_n^{BM}(X, A) \to E^2_{n-2t+j+1,2t-j-1}$$

for each $j \geq 0$. It therefore suffices to prove $E^2_{p,q} = 0$ whenever the inequalities $p+q \geq d+t$ and $q < 2t$ hold. Observe that these inequalities imply $q > 2d-2p$, and so it suffices to show $E^2_{p,q} = 0$ whenever $q > 2d - 2p$. These E_2-terms can be given as

$$E^2_{p,q} = h_p(H_q(X_d^{an}, A) \to \cdots \to H_q(X_0^{an}, A))$$

where X_i is a smooth, projective complex variety of dimension at most $d - i$ [21, Theorem 2]. Since $H_q(X_i^{an}, A) = 0$ if $q > 2d - 2i$, we have that $E^2_{p,q} = 0$ if $q > 2d - 2p$, as desired. □

Motivated by the above theorem, we make the following definition:

Definition 6.5. We define $\mathcal{C}$ to be the class of objects X in $Sch/\mathbb{C}$ satisfying the condition that the natural map of (5.12)

$$L_t H_n(X) \to \tilde{W}_{-2t} H_n^{BM}(X)$$

is an isomorphism for all t and n.

Observe that by Theorem 6.3, if X belongs to $\mathcal{C}$, then $K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$ is an isomorphism for $i \geq \dim(X) - 1$ and a monomorphism for $i = \dim(X) - 2$.

Remark 6.6. Since the groups $\tilde{W}_t H_n^{BM}$ are always finitely generated, the Lawson homology of varieties in $\mathcal{C}$ is finitely generated. In particular, if X is smooth and in $\mathcal{C}$ then the semi-topological spectral sequence implies that its semi-topological K-theory is finitely generated in each degree.

We give some initial examples of varieties in this class:

Example 6.7. For any natural number n, the affine space $\mathbb{A}^n$ is in $\mathcal{C}$.

Example 6.8. Any smooth projective curve is in $\mathcal{C}$.

Indeed, in both these examples, the weights are pure, so being in $\mathcal{C}$ is equivalent to the fact that the generalized cycle map $L_t H_n \to H_n^{BM}$ is an isomorphism for $n \geq 2t$. In the case of affine space, that follows from duality and homotopy invariance for morphic cohomology, and for curves it is trivial.

The next collection of examples is built of these basic ones using certain constructions under which the class $\mathcal{C}$ is closed.

Proposition 6.9. *The class $\mathcal{C}$ is closed under the following constructions.*

(1) *Closure under localization: Let $Z \subset X$ be a closed immersion with Zariski open complement $U = X - Z$. Then if two of X, Z, and U belong to $\mathcal{C}$, then so does the third.*
(2) *Closure for bundles: For a vector bundle $E \to X$, the variety X belongs to $\mathcal{C}$ if and only if $\mathbb{P}(E)$ does. In this case, E belongs to $\mathcal{C}$ as well.*
(3) *Closure under blow-ups: Let $Z \subset X$ be a regular closed immersion and such that Z belongs to $\mathcal{C}$. Then X is in $\mathcal{C}$ if and only if the blow-up X_Z of X along Z is in $\mathcal{C}$.*

Proof. Claim (1) follows from Theorem 5.12 and the five lemma.

The first part of (2) follows directly from Theorem 5.14. In this situation, both $\mathbb{P}(E \oplus 1)$ and $\mathbb{P}(1)$ are also in $\mathcal{C}$ and thus so is $E = \mathbb{P}(E \oplus 1) - \mathbb{P}(1)$ by (1).

To prove (3), note that $Z' = X_Z \times_X Z \to Z$ is a projectivized vector bundle (since $Z \rightarrowtail X$ is a regular closed immersion) and hence Z' also belongs to $\mathcal{C}$ by (2). Since $X_Z - Z' = X - Z$, the result now follows from claim (1). □

A large class of varieties that is contained in $\mathcal{C}$ is the class of linear varieties that has been introduced by Jannsen [24] as a class of varieties for which Tate's conjecture on algebraic cycles can be proved.

Definition 6.10. The class of *linear* varieties is the smallest class of quasi-projective complex varieties satisfying the conditions:

- All affine spaces are linear.
- if X is a quasi-projective complex variety, $Z \subset X$ is a closed subscheme, $U = X - Z$ is the open complement, Z and either U or X are linear, then so is the remaining member of the triple.

Proposition 6.11. *All linear varieties are in* $\mathcal{C}$.

Proof. This follows immediately from Example 6.7 and localization. □

Using different techniques than ours, R. Joshua [25] has proven that if X is a smooth, projective linear variety, then the motivic cohomology with finite coefficients (which is equal to morphic cohomology with finite coefficients) of X is isomorphic to the singular cohomology with finite coefficients of X^{an}. Proposition 6.11 thus reproduces and extends Joshua's result for complex varieties. (Joshua also considers varieties over an arbitrary algebraically closed field.)

A simple induction argument shows that the Chow groups $CH_*(X)$ of a linear variety X are finitely generated, and hence that $CH_t(X) \cong L_t H_{2t}(X)$, for all $t \geq 0$, since the kernel of $CH_t(X) \to L_t H_{2t}(X)$ is divisible. Using Theorem 5.11 (3) and Proposition 6.11, we deduce the following slight improvement of a result of B. Totaro [38, Theorem 3]. (Note that Totaro's definition of "linear variety" is slightly more restrictive than ours.)

Theorem 6.12. *For a linear variety X, the natural map induces a rational isomorphism*

$$CH_t(X)_{\mathbb{Q}} \cong W_{-2t} H_{2t}^{BM}(X^{an}, \mathbb{Q}),$$

for all $t \geq 0$.

More generally, the map $CH_t(X)_{\mathbb{Q}} \to W_{-2t} H_{2t}^{BM}(X^{an}, \mathbb{Q})$ is surjective for any X in the class $\mathcal{C}$, but it usually fails to be injective (for example, it is not injective for curves).

Example 6.13. The following varieties are examples of linear varieties:

(1) Projective spaces.
(2) More generally, cellular varieties — that is, varieties X such that there is a chain of closed immersions $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_m = X$ such that $X_i - X_{i-1}$ is an affine space for all $i = 0, \ldots, m$. Examples of cellular varieties are flag varieties.

(3) Every product $\mathbb{G}_m^{\times n}$ of multiplicative groups.
(4) Toric varieties.

The fact that projective cellular varieties are in $\mathcal{C}$ has already been shown by Lima-Filho [28]. The first three of these examples are obviously linear. For the last one, we use that any toric variety X admits a filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_d = X$ by closed subschemes such that $X_i - X_{i-1}$ is a disjoint union of i-dimensional tori. (Specifically, one takes X_i to be the closure of the i-dimensional orbits — see [20, §3]).

Example 6.14. More generally, if X is any variety in $\mathcal{C}$, $T \to X$ is a principal $\mathbb{G}_m^{\times n}$-bundle and V is a toric variety with dense orbit $\mathbb{G}_m^{\times n}$, then the "toric fibration" $T \times_X V$ is also in $\mathcal{C}$.

Indeed, this follows from (1) and (2) of Proposition 6.9: Observe that any principal torus bundle is locally trivial in the Zariski topology (by Hilbert's Theorem 90). We argue as in Example 6.13 (4): The toric fibration $T \times_X V$ has a filtration (induced by that of V) by closed subschemes X_i such that $X_i - X_{i-1}$ is a disjoint union of torus bundles (locally trivial in the Zariski topology) over X. This implies that $X_i - X_{i-1}$ is in $\mathcal{C}$, because any torus bundle is the open complement of the zero section of a vector bundle over X.

For other computations of cohomology and K-theory of toric fibrations, see [32].

Example 6.15. All schemes of dimension at most 1 are in $\mathcal{C}$.

This follows from (1) of Proposition 6.9, since smooth projective curves and the point are in $\mathcal{C}$.

Example 6.16. A smooth, quasi-projective complex surface S belongs to $\mathcal{C}$ if and only if it is birationally equivalent to a smooth, projective surface $\overline{S}$ such that all elements of $H^2(\overline{S}^{an})$ are algebraic — i.e., if and only if the injective map $L^1H^2(\overline{S}) \to H^2(\overline{S}^{an})$ is actually a surjection for some such $\overline{S}$. In particular, all rational surfaces belong to $\mathcal{C}$.

Indeed, by Proposition 6.9 and Example 6.15, S belongs to $\mathcal{C}$ if and only if $\overline{S}$ does, and as in the proof of Theorem 3.7, computations in Lawson homology (including especially [12, 9.3]) show that $L_tH_n(\overline{S}) \to \tilde{W}_{-2t}H_n(\overline{S})$ is an isomorphism except possibly when $t = 1$ and $n = 2$, in which case it is injective. (Recall that $\tilde{W}_{-2t}H_n(\overline{S})$ is $H_n(\overline{S})$ for $n \geq 2t$ and 0 otherwise.)

Observe that most surfaces do not satisfy the condition of Example 6.16. Indeed, if S is smooth and projective, then in the Hodge decomposition of the complex cohomology of S^{an}

$$H^2(S^{an}, \mathbb{C}) = H^{2,0}(S^{an}, \mathbb{C}) \oplus H^{1,1}(S^{an}, \mathbb{C}) \oplus H^{0,2}(S^{an}, \mathbb{C}),$$

only $H^{1,1}(S^{an}, \mathbb{C})$ can be algebraic. Thus if S^{an} admits a global holomorphic 2-form, then $H^{2,0}(S^{an}, \mathbb{C}) = H^0(S^{an}, \Omega^2) \neq 0$ and hence S does not belong to

$\mathcal{C}$. For example, if S is a product of two smooth, projective curves having positive genera, then $S \notin \mathcal{C}$. However, all smooth surfaces do nevertheless satisfy the conclusion of Theorem 7.14 below.

The next result is, in some sense, a generalization of the assertion, proven in [4], that any complex vector bundle on a smooth projective rational three-fold is algebraic.

Proposition 6.17. *Let X be a smooth projective rational three-fold. Then X is in $\mathcal{C}$.*

Proof. By the factorization result of [1], we can factor the birational transformation $X \dashrightarrow \mathbb{P}^3$ into a sequence of transformations

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = \mathbb{P}^3$$

such that each transformation $X_i \dashrightarrow X_{i+1}$ is either a blow-up along a smooth center or the (rational) inverse of such a blow-up. Since any nontrivial blow-up of a smooth three-fold along a smooth subvariety has center of dimension at most 1, Proposition 6.9 (3) and Example 6.15 show that X_i is in $\mathcal{C}$ if and only if X_{i+1} is in $\mathcal{C}$, for all $i \in 0, ..., n-1$. Since $\mathbb{P}^3$ is in $\mathcal{C}$ by Example 6.13, we conclude that X is in $\mathcal{C}$, as claimed. □

Observe that the proof of Proposition 6.17 shows that belonging to $\mathcal{C}$ is a birational invariant for smooth, projective three-folds.

To conclude, we have now proven the following:

Theorem 6.18. *Let X be one of the following complex varieties:*

(1) *A smooth quasi-projective curve.*
(2) *A smooth, quasi-projective surface having a smooth compactification with all of H^2 algebraic.*
(3) *A smooth projective rational three-fold.*
(4) *A smooth quasi-projective linear variety (e.g., a smooth quasi-projective toric variety).*
(5) *A smooth toric fibration over one of the above.*

Then for any abelian group A the natural map $K_n^{sst}(X, A) \to ku^{-n}(X^{an}, A)$ is an isomorphism for $n \geq \dim(X) - 1$ and a monomorphism for $n = \dim(X) - 2$. If X is projective, this map is an isomorphism for all $n \geq 0$.

For four-folds, we can prove the following result:

Proposition 6.19. *Let X be a smooth projective rational four-fold. Then for any abelian group A the natural map $K_n^{sst}(X, A) \to ku^{-n}(X^{an}, A)$ is an isomorphism for $n \geq 1$ and a monomorphism for $n = 0$.*

Proof. Thomason's computation of the K-theory of a blow-up with regularly embedded center (cf. [36]) in the form of Gillet-Soulé (cf. [21, Theorem 5]) implies that for any regular embedding $Y \to X$ of Noetherian schemes, the square of spectra

$$\begin{array}{ccc} \mathcal{K}(Y') & \longleftarrow & \mathcal{K}(X') \\ \uparrow & & \uparrow \\ \mathcal{K}(Y) & \longleftarrow & \mathcal{K}(X) \end{array}$$

is naturally split homotopy cartesian, where X' is the blow-up of X along Y and Y' is the exceptional divisor of that blow-up; here "split" means that there is a section $\mathcal{K}(Y') \to \mathcal{K}(Y)$ of the pull-back map. Consequently, the corresponding diagram for singular semi-topological K-theory

$$\begin{array}{ccc} \mathcal{K}^{sst}(Y') & \longleftarrow & \mathcal{K}^{sst}(X') \\ \uparrow & & \uparrow \\ \mathcal{K}^{sst}(Y) & \longleftarrow & \mathcal{K}^{sst}(X) \end{array}$$

will also be split homotopy cartesian. That is, there are natural isomorphisms

$$K_n^{sst}(X') \cong K_n^{sst}(X) \oplus \bigoplus_{i=1}^{c-1} K_n^{sst}(Y) \tag{6.20}$$

for any $n \geq 0$, with c the codimension of the embedding. On the other hand, excision shows that the same is true for topological K-theory.

Assume Y and X are smooth varieties and $\dim(Y) \leq 2$. Using Theorem 3.7 and Example 6.15, we conclude that $K_n^{sst}(X) \to ku^{-n}(X^{an})$ is an isomorphism for $n \geq 1$ and a monomorphism for $n = 0$ if and only if the same holds for $K_n^{sst}(X') \to ku^{-n}(X'^{an})$. Now let X be a smooth rational projective fourfold. The birational transformation $X \dashrightarrow \mathbb{P}^4$ can be factored into a sequence of blowings-up and blowings-down with smooth centers (cf. [1]) which will be of dimension at most 2; consequently Example 6.13 (1) implies our assertion. □

7. Conjectures in morphic cohomology and K-theory

In this section, we state semi-topological analogues of the usual conjectures relating motivic cohomology, étale cohomology, algebraic K-theory, and topological K-theory, and we explain the connections between such conjectures. Contrary to the algebraic conjectures, which are about cohomology or K-theory with finite coefficients, these conjectures are *integral*. The main result of this section is Theorem 7.3 which allows us to state the morphic cohomology analogue of the Beilinson-Lichtenbaum conjecture as suggested by A. Suslin.

Let $\pi : (Sch/\mathbb{C})_{et} \to (Sch/\mathbb{C})_{Zar}$ be the morphism of sites from the étale site to the Zariski site. For any $q \geq 0$, let $\mathbb{Z}/m(q)$ denote the weight q motivic complex with $\mathbb{Z}/m$ coefficients, defined for example in [35], so that we have the identity

$$H^n_{\mathcal{M}}(X, \mathbb{Z}/m(q)) = H^n_{Zar}(X, \mathbb{Z}/m(q)).$$

As shown in [35], for any positive integer m, there is a canonical map

$$\mathbb{Z}/m(q) \to tr_{\leq q}\mathbb{R}\pi_*\mathbb{Z}/m \tag{7.1}$$

of complexes of sheaves on $(Sch/\mathbb{C})_{Zar}$ from the motivic complex to the good truncation of the total right derived functor of π_* applied to the constant sheaf $\mathbb{Z}/m$ on $(Sch/\mathbb{C})_{et}$. The *Beilinson-Lichtenbaum conjecture* (for complex varieties) asserts that the homomorphism (7.1) is a quasi-isomorphism for smooth, quasi-projective complex varieties.

Since it is known that $H^n_{\mathcal{M}}(-, \mathbb{Z}/m(q))$ vanishes locally for the Zariski topology if $n > q$, the Beilinson-Lichtenbaum Conjecture is equivalent to the assertion that, for any smooth, quasi-projective complex variety X and integer $m \geq 1$, the canonical map

$$H^n_{\mathcal{M}}(X, \mathbb{Z}/m(q)) \to H^n_{et}(X, \mathbb{Z}/m)$$

is an isomorphism for $n \leq q$ and a monomorphism for $n = q+1$. Since étale cohomology and singular cohomology agree for finite coefficients, this statement is in turn equivalent to the assertion that

$$H^n_{\mathcal{M}}(X, \mathbb{Z}/m(q)) \to H^n(X^{an}, \mathbb{Z}/m) \tag{7.2}$$

is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$. This reformulation of the Beilinson-Lichtenbaum Conjecture has the advantage that it can be verified on a case-by-case basis for certain varieties. Therefore we say that the *Beilinson-Lichtenbaum Conjecture holds for a smooth variety X* if the map (7.2) is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$.

In order to state Conjecture 7.8 (the semi-topological analogue of the Beilinson-Lichtenbaum conjecture), we first need to prove the existence of the corresponding homomorphism of complexes of sheaves analogous to the homomorphism (7.1).

Recall that the weight t morphic cohomology of a complex variety can be given as the hypercohomology (in the Zariski topology) of the complex of abelian sheaves

$$\mathbb{Z}(q)^{sst} = \mathrm{Hom}(- \times \Delta^\bullet_{top}, C_0(\mathbb{P}^q))^+/\,\mathrm{Hom}(- \times \Delta^\bullet_{top}, C_0(\mathbb{P}^{q-1}))^+[-2q],$$

where $C_0(\mathbb{P}^q)$ is the infinite disjoint union of of quasi-projective varieties parameterizing effective zero cycles on $\mathbb{P}^q$. That is, $C_0(\mathbb{P}^q) = \coprod_n Symm^n(\mathbb{P}^q)$, and from this we deduce the existence of a map of sheaves on $\underline{CW}$

$$\epsilon^*\mathbb{Z}(q)^{sst} \to \mathrm{Maps}(- \times \Delta^\bullet_{top}, (\coprod_n Symm(S^{2q}))^+)[-2q].$$

(see Notation 2.2 for the meaning of $\underline{CW}$ and ϵ). The Dold-Thom Theorem shows that $(\coprod_n Symm^n(S^{2q}))^+$ is a $K(\mathbb{Z}, 2q)$ and thus we obtain a canonical map $\epsilon^*\mathbb{Z}(q)^{sst} \to \mathbb{Z}$. By adjointness, we obtain the map

$$\mathbb{Z}(q)^{sst} \to \mathbb{R}\epsilon_*\mathbb{Z}.$$

of complexes of sheaves on $(Sch/\mathbb{C})_{Zar}$.

Theorem 7.3. *The complex of sheaves $\mathbb{Z}(q)^{sst}$ on $Sm/\mathbb{C}$ has no cohomology in degrees greater than q, so that the canonical map $tr_{\leq q}\mathbb{Z}(q)^{sst} \to \mathbb{Z}(q)^{sst}$ is a quasi-isomorphism. Consequently, the map $\mathbb{Z}(q)^{sst} \to \mathbb{R}\epsilon_*\mathbb{Z}$ factors in the derived category as*

$$\mathbb{Z}(q)^{sst} \to tr_{\leq q}\mathbb{R}\epsilon_*\mathbb{Z} \tag{7.4}$$

followed by the canonical map $tr_{\leq q}\mathbb{R}\epsilon_*\mathbb{Z} \to \mathbb{R}\epsilon_*\mathbb{Z}$.

More generally, for any abelian group A, the complex $A(q)^{sst} = \mathbb{Z}(q)^{sst} \otimes_{\mathbb{Z}} A$ has no cohomology in degrees greater than q so that we have an induced map in the derived category

$$A(q)^{sst} \to tr_{\leq q}\mathbb{R}\epsilon_*A. \tag{7.5}$$

Proof. We need to prove the presheaf $L^qH^n(-; A)$ vanishes locally on any smooth, quasi-projective complex variety for all $n > q$. It suffices to prove the result when A is finitely generated, and so we may take $A = \mathbb{Z}/m$ or $A = \mathbb{Z}$. This result is known when $A = \mathbb{Z}/m$ for any positive integer m, since in this case $L^qH^n(-; \mathbb{Z}/m)$ is naturally isomorphic to $H^n_{\mathcal{M}}(-; \mathbb{Z}/m(q))$. For the case $A = \mathbb{Z}$, a torsion element of $L^qH^n(X, \mathbb{Z})$, for a smooth variety X, must vanish locally if $n > q + 1$. If $n = q + 1$, then a torsion element α of $L^qH^{q+1}(X, \mathbb{Z})$ lifts (locally on X) to a class in $L^qH^q(X, \mathbb{Z}/m) \cong H^q_{\mathcal{M}}(X, \mathbb{Z}/m(q))$ for some $m > 0$. Since $H^q_{\mathcal{M}}(-, \mathbb{Z}(q)) \to H^q_{\mathcal{M}}(-, \mathbb{Z}/m(q))$ is locally surjective (due to the local vanishing of $H^{q+1}_{\mathcal{M}}(-, \mathbb{Z}(q))$), so is $L^qH^q(-, \mathbb{Z}) \to L^qH^q(-, \mathbb{Z}/m)$. It follows that α is locally trivial. Consequently, it remains to prove the result in the case $A = \mathbb{Q}$.

Since the presheaf $L^qH^n(-; \mathbb{Q})$ is a pre-theory, it suffices by [39] to prove $L^qH^n(-; \mathbb{Q})$ vanishes for $n > q$ at the generic point on any smooth, connected, quasi-projective complex variety X. By using duality for Lawson homology and morphic cohomology, it suffices to prove $L_tH_m(-; \mathbb{Q})$ vanishes at the generic point of X for $m < t + d$, where $d = \dim(X)$. By choosing a smooth, projective compactification of X, we may assume X is projective. Recall that we have $L_tH_m(X; \mathbb{Q}) = \pi_{m-2t}(Z_t(X), \mathbb{Q})$ where $Z_t(X)$ denotes the space of t-dimensional algebraic cycles on X and for any open subvariety U of X with closed complement Y, we have $L_tH_m(U; \mathbb{Q}) = \pi_{m-2t}(Z_t(U), \mathbb{Q})$ where $Z_t(U) =$

$Z_t(X)/Z_t(Y)$. We claim that to prove the vanishing of $L_t H_m(-;\mathbb{Q})$ near the generic point of X for $m < t + d$, it suffices to show the canonical map

$$\varinjlim_{Y \subset X} H_n(Z_t(Y)) \to H_n(Z_t(X)) \tag{7.6}$$

is an isomorphism for $n < d - t - 1$ and surjective for $n = d - t - 1$, where Y ranges over all closed subschemes of X of codimension one. For suppose (7.6) were an isomorphism for $n < d - t - 1$ and surjective for $n = d - t - 1$. Since $Z_t(X)$ and $Z_t(Y)$, for each Y, are topological abelian groups, the rational Hurewicz homomorphisms are injective, and thus

$$0 \to \varinjlim \pi_n(Z_t(Y), \mathbb{Q}) \to \pi_n(Z_t(X), \mathbb{Q}) \to \varinjlim \pi_n(Z_t(X)/Z_t(Y), \mathbb{Q}) \to 0$$

would be a short exact sequence for $n < d - t$. Since

$$\varinjlim_{Y} \pi_n(Z_t(X)/Z_t(Y), \mathbb{Q}) \to \varinjlim_{Y} H_n(Z_t(X)/Z_t(Y); \mathbb{Q})$$

is also injective and the map $H_n(Z_t(X), \mathbb{Q}) \to \varinjlim_{Y} H_n(Z_t(X)/Z_t(Y); \mathbb{Q})$ factors through $\varinjlim_{Y} H_n(Z_t(X), Z_t(Y); \mathbb{Q})$, the map

$$\pi_n(Z_t(X), \mathbb{Q}) \to \varinjlim_{Y} \pi_n(Z_t(X)/Z_t(Y), \mathbb{Q})$$

would be the zero map, and hence

$$\varinjlim_{Y} \pi_n(Z_t(X)/Z_t(Y), \mathbb{Q}) = \varinjlim_{Y} L_t H_{n+2t}(X - Y, \mathbb{Q})$$

would vanish for $n < d - t$.

It remains to prove (7.6) is an isomorphism for $n < d - t - 1$ and surjective for $n = d - t - 1$. Since $Z_t(X)$ is the homotopy theoretic group completion of $C_t(X) = \coprod_e C_{t,e}(X)$ (where $C_{t,e}(X)$ denotes the projective variety parameterizing effective cycles of dimension t and degree e) we have that the homology of $Z_t(X)$ is obtained from the homology of $C_t(X)$ by inverting the action of the abelian monoid $\pi_0 C_t(X)$. Similarly, the group $\varinjlim_{Y} H_n(Z_t(Y))$ is obtained from $\varinjlim_{Y} H_n(C_t(Y))$ by inverting the action of $\varinjlim_{Y} \pi_0 C_t(Y)$. Since the map

$$\varinjlim_{Y} \pi_0 C_t(Y) \to \pi_0 C_t(X)$$

will be an isomorphism provided the analogous map of H_0 groups is, we see that it suffices to prove

$$\varinjlim_{Y} H_n(C_{t,e}(Y)) \to H_n(C_{t,e}(X)) \tag{7.7}$$

is an isomorphism for $n < d - t - 1$ (resp., an epimorphism for $n = d - t - 1$), for all $e \geq 0$.

To establish that (7.7) is an isomorphism for $n < d-t-1$ (resp., epimorphism for $n = d-t-1$), we will use the singular Lefschetz theorem of [2], which states that given a (possibly singular) projective variety P, the map $H_n(W) \to H_n(P)$ is an isomorphism (resp., epimorphism) if $W = H_1 \cap \cdots \cap H_j \subset P$ is a complete intersection of dimension at least $n+1$ (resp., of dimension at least n), where H_1 is a hypersurface of P containing the singular locus of P, and, for all $i \geq 2$, H_i is a hypersurface in $H_1 \cap \cdots \cap H_{i-1}$ containing the singular locus of $H_1 \cap \cdots \cap H_{i-1}$.

Assume $n < d - t$. Let $I \subset C_{t,e}(X) \times X$ be the incidence variety such that the reduced fiber of $I \to C_{t,e}(X)$ over an effective t-cycle Z is $|Z|$, the support of the effective cycle Z. In particular, $I \to C_{t,e}(X)$ is equidimensional of relative dimension t. Given a class $a \in H_n(C_{t,e}(X))$, by the singular Lefschetz theorem, there is a reduced closed subscheme $W \subset C_{t,e}(X)$ of dimension n such that a lifts to $H_n(W)$. Define the "support" of W to be $supp(W) = \pi_2(\pi_1^{-1}(W))$, where $\pi_1 : I \to C_{t,e}(X)$ and $\pi_2 : I \to X$ are the evident projection maps. Since π_1 is equidimensional of relative dimension t, we have that $supp(W)$ is a closed subscheme of X having dimension at most $n + t < d$. Observe that $W \subset C_{t,e}(supp(W))$, and so a lifts to $H_n(C_{t,e}(supp(W))$ and hence to $\varinjlim_Y H_n(C_{t,e}(Y))$ as desired.

Now assume that $n < d - t - 1$. To prove that (7.7) is injective, suppose $a \in H_n(C_{t,e}(Y))$ maps to 0 in $H_n(C_{t,e}(X))$, where Y is a proper closed subscheme of X. It suffices to show there is another proper closed subscheme Y' with $Y \subset Y' \subset X$ such that a maps to 0 in $H_n(C_{t,e}(Y'))$. By the weak Lefschetz applied to $C_{t,e}(Y)$, there is a complete intersection $W \subset C_{t,e}(Y)$ having dimension at most $n+1$ such that $H_n(W) \to H_n(C_{t,e}(Y))$ is an isomorphism. Since $C_{t,e}(Y)$ is a closed subscheme of $C_{t,e}(X)$, we can find a complete intersection $W' \subset C_{t,e}(X)$ such that $W \subset W'$, $dim(W') = n+1$, and $H_n(W') \to H_n(C_{r,e}(X))$ is an isomorphism. (Specifically, form $W' = H_1 \cap \cdots \cap H_j$ by taking H_i to be a hypersurface of $H_1 \cap \cdots \cap H_{i-1}$ containing both the singular locus and W.) In particular, the image of a_W in $H_n(W')$ vanishes. Finally, let $W'' = W' \cup C_{t,e}(Y)$. Observe that $supp(W'') = supp(W') \cup supp(C_{t,e}(Y)) = supp(W') \cup Y$ is a closed subscheme of X that contains Y and has dimension at most $t + n + 1 < d$. We set $Y' = supp(W'')$. Then the image of a in $H_n(C_{t,e}(Y'))$ vanishes, since $C_{t,e}(Y')$ clearly contains W'' and hence W'. □

The following conjecture is due to A. Suslin:

Conjecture 7.8 (Suslin's Conjecture). *The map (7.4) is a quasi-isomorphism after restricting it to smooth varieties, and hence for any smooth, quasi-projective complex variety X, it induces an isomorphism*

$$L^q H^n(X) = H^n_{Zar}(X, \mathbb{Z}^{sst}(q)) \cong H^n_{Zar}(X, tr_{\leq q}\mathbb{R}\epsilon_*\mathbb{Z}).$$

More generally, for any abelian group A, the map (7.5) induces an isomorphism

$$L^q H^n(X, A) \cong H^n_{Zar}(X, tr_{\leq q}\mathbb{R}\epsilon_* A),$$

for all such X.

As before, it is useful to establish an equivalent formulation of Suslin's Conjecture that allows for verification on a case-by-case basis.

Proposition 7.9. *Suslin's Conjecture with coefficients in A is equivalent to the assertion that, for any smooth, quasi-projective complex variety X, the canonical map*

$$L^q H^n(X, A) \to H^n(X^{an}, A) \tag{7.10}$$

is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$.

Proof. The fact that Suslin's Conjecture implies such isomorphisms and monomorphisms is obvious. Conversely, assume that for all smooth, quasi-projective varieties X, the map $L^q H^n(X, A) \to H^n(X^{an}, A)$ is an isomorphism for $n \leq q$. Then, in particular, such an isomorphism holds locally on any smooth, quasi-projective variety X, and so the natural map $tr_{\leq q} A(q)^{sst} \to tr_{\leq q}\mathbb{R}\epsilon_* A$ is a quasi-isomorphism of complexes of Zariski sheaves. Now Suslin's Conjecture follows from Theorem 7.3. □

Remark 7.11.

(1) Similar to the Beilinson-Lichtenbaum Conjecture, we say that *Suslin's Conjecture (with coefficients in A) holds for a smooth variety X* if the map 7.10 is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$.
(2) Observe that Suslin's Conjecture implies the vanishing of $L^t H^n(X)$ for all smooth varieties X whenever $n < 0$ (equivalently, the vanishing of $L_s H_m(X)$ for $m > 2\dim(X)$). For smooth varieties of dimension ≥ 3, such vanishing (conjectured in [13] and closely related to the Beilinson-Soule vanishing conjecture [5]) is known only for very special varieties such as those of Theorem 6.17. For example, such vanishing is not even known for the product of three elliptic curves.

Recall that the Quillen-Lichtenbaum Conjecture for complex varieties asserts the following: For a smooth, quasi-projective complex variety X and integer $m \geq 1$, the canonical map

$$K_n(X, \mathbb{Z}/m) \to ku^{-n}(X^{an}, \mathbb{Z}/m) \tag{7.12}$$

is an isomorphism for $n \geq \dim(X) - 1$ and a monomorphism for $n = \dim(X) - 2$. The semi-topological analogue of the Quillen-Lichtenbaum conjecture involves integral coefficients:

Conjecture 7.13 (Semi-topological Quillen-Lichtenbaum). *For a smooth, quasi-projective complex variety X the canonical map*

$$K_n^{sst}(X) \to ku^{-n}(X^{an})$$

is an isomorphism for $n \geq \dim(X)-1$ and a monomorphism for $n = \dim(X)-2$. More generally, such an isomorphism holds in the given range with coefficients in any abelian group A.

Theorem 7.14. *The Beilinson-Lichtenbaum, Suslin, Quillen-Lichtenbaum, and semi-topological Quillen-Lichtenbaum Conjectures hold for the following complex varieties:*

(1) *smooth quasi-projective curves,*
(2) *smooth quasi-projective surfaces,*
(3) *smooth projective rational three-folds,*
(4) *smooth quasi-projective toric varieties,*
(5) *smooth toric fibrations over varieties of type (1), (3), and (4) and over smooth, quasi-projective surfaces having smooth compactifications with all of H^2 algebraic.*

Proof. That these varieties satisfy the semi-topological Quillen-Lichtenbaum Conjecture (with arbitrary coefficients) follows from Theorem 6.18 and Theorem 3.7. The proof of Theorem 6.3 shows that a variety X in $\mathcal{C}$ satisfies the condition that $L_t H_n(X, A) \to H_n^{BM}(X, A)$ is an isomorphism for $n \geq d + t$ and a monomorphism for $n = d + t - 1$, for any abelian group A. When X is smooth and in $\mathcal{C}$, duality then implies that Suslin's Conjecture holds for X with coefficients in A. The proof of Theorem 3.7 shows that smooth, quasi-projective surfaces satisfy Suslin's Conjecture. Finally, the classical Quillen-Lichtenbaum Conjecture (resp., the Beilinson-Lichtenbaum Conjecture) holds for these varieties, since semi-topological K-theory (resp., morphic cohomology) with finite coefficients coincides with algebraic K-theory (resp., motivic cohomology). □

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